Mathematics G3 for Mechanical Engineers (BMETE93BG03)

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Prerequisites: Mathematics G1, G2.

Program:

Classification of differential equations. Separable ordinary differential equations, linear equations with constant and variable coefficients, systems of linear differential equations with constant coefficients. Some applications of ODEs. Scalar and vector fields. Line and surface integrals. Divergence and curl, theorems of Gauss and Stokes, Green formulae. Conservative vector fields, potentials. Some applications of vector analysis. (4 hours/4 credits)

Literature:

Thomas' Calculus by Thomas, G.B. et al. Addison-Wesley, Several editions (ISBN0321185587)

Detailed program (Spring 2025)

Week 1.

Lecture 1. (February 11, Tuesday).

Ordinary differential equations are mathematical models for time dependent finite dimensional differentiable deterministic processes.

First order ordinary differential equations, ODEs, $y' = \frac{dy}{dx} = f(x,y)$. Definition of the solution, integral curve, trajectory. Initial value (Cauchy) problem: $y' = f(x,y).\, y(x_0) = y_0$. Theorem of Existence & Uniqueness for the initial value problem (without proof) with conditions $f, \frac{\partial f}{\partial y} \in C^0(D)$.

Examples (guess the solutions). General solution. . Initial value problems $y' = y, y(0) = 1$, $y = e^x$. Maximal interval is "smaller".$y' = 1 + y^2, y(0) = 0, y = \tan x$. Checking the solution: $y' = 1 + x^2, y(0) = 0, y = x + \frac{x^3}{3}$. Remark (in general): $y' = f(x)$, $y = \int f(x)dx + c$.

There is no uniqueness $y' = 3y^{\frac{2}{3}}, y(0) = 0$.

Special cases. Separable equations. $\frac{dy}{dx} = g(x)H(y)$. (i) $H(y) = 0$, constant solutions. (ii) $\int \frac{dy}{H(y)} = \int g(x)dx + c$ (with proof in case of initial value problem).

Example (trivial): $y' = y$, $(i)\ y(0) = 1, (ii)\ y(0) = 0$. Solution (i) $y(x) = e^x$, (ii) $y(x) = 0$. $y' = 1 + y^2, y(0) = 0$, $y = \tan x$ (solve as separable equation). $y' = 2x\, cos^2\, y$, $(i)\ y(0) = \frac{\pi}{2}, (ii)\ y(0) = 0$. Solutions$(i)\ y = \frac{\pi}{2},\ (ii)\ y = arctan\, x^2$.

Lecture 2. (February 12, 2025, Wednesday).

Special case. Separable equation. Example (non-trivial) $y' = 2x\, cos^2\, y$, $(i)\ y(0) = -\frac{\pi}{2}, (ii)\ y(0) = 0$. Solutions$(i)\ y = \frac{\pi}{-2},\ (ii)\ y = arctan\, x^2$.

Another special case: Linear equation. $y' + P(x)y = Q(x)$. Solution:

$y = ce^{\int -P(x)dx} + e^{\int -P(x)dx} \int e^{\int P(x)dx} Q(x)dx$ (proof by the variation of constants).

Examples (solving linear equations – general solution, initial value problem, sketching the integral curve(s): $y' = y$, $(i)\ y(0) = 1$, $(ii)\ y(0) = 0$. Solution (i) $y(x) = e^x$, (ii) $y(x) = 0$. $xy' = 3y + x^2$, $y(1) = 0$. Solution $y = x^3 - x^2$. $0 < x$.

Another exemple (homework). $y' \cos x + y \sin x - 1 = 0$, $(i)\ y(0) = 0$, $(ii)\ y(0) = 1$).
Solutions (i) $y = \sin x$, (ii) $y = \cos x + \sin x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Literature: Thomas' Calculus Chapter 9, § 1,2,5.

Homework (optional): Thomas' Calculus. **9.1**.11,17, **9.2**.15.

Remark: $y' + 2y = 3, y(0) = 1$ is both separable and linear equation.

Week 2.

Lecture 3. (February 18, Tuesday).

Orthogonal trajectories. Theory. Examples: Straight lines passing through the origin, concentric circles centered at the origin – solution in both directions. Cartesian, polar coordinates ($y = cx$, $x^2 + y^2 = c^2$).
Further example (homework): orthogonal trajectories $y = ce^x$, $y = \ln(cx)$. Orthogonal trajectories – parabolas.

Lecture 4. (February 19, Wednesday)
First order autonomous differential equation $y' = f(y)$. Phase portrait (line, trajectories).

Sketching the graph of some typical integral curves (with inflection points). Examples. , $y' = y^2 - 2y$. Further example (homework): $y' = \sin y$.

ODEs of the second order. $y'' = f(x, y, y')$, $y(x_0) = y_0$, $y'(x_0) = y_0'$. Theorem of existence and uniqueness.

Literature: Thomas' Calculus Chapter **9, § 3, 4, 5**.

Week 3.

Lecture 5. (February 25, Tuesday).

ODEs of the second order. Examples (solution by 2 methods – elementary ideas, later linear equations with constant coefficients): $my'' = mg$ – gravity, $y'' + y = 0$ – harmonic oscillator.

Linear second order equations. Theorem of structure (general solution of non-homogeneous equation = general solution of homogeneous equation + particular solution of non-homogeneous equation.

Linear equations with constant coefficients. General solution of homogeneous equation (3 cases). Particular solution of nonhomogeneous linear equations with constant coefficients.in special case of polynomial, exponential trigonometric right-hand sides.

Solve the initial value problem $y'' + y' = -1,\ y(0) = 1,\ y'(0) = -2$ by 3 methods (linear equation, reducible equation, Newton's method – until the third order terms). The idea of these methods: (i) Linear with constant coefficients. Homogeneous, characteristic equation, nonhomogeneous, resonance. (ii) Substitution $u(x) = y'(x)$. (iii) Newton's method. We are looking for the solution in form $y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \cdots = 1 - 2x + \frac{x^2}{2} + \cdots$. Only (i) was solved at this lecture.

Lecture 6. (February 26, Wednesday)

Reducible by substitution to 1st order equations 2nd order equations:
y=f(x,y'), u(x)=y'(x), $y = f(y, y')$, u(y)=y'(x) .

Continuation of the previous exercise – cases (ii), (iii). Solve the initial value problem $y'' + y' = -1,\ y(0) = 1,\ y'(0) = -2$ by 3 methods (linear equation, reducible equation, Newton's method – until the third order terms). The idea of these methods: (i) Linear with constant coefficients. Homogeneous, characteristic equation, nonhomogeneous, resonance. (ii) Substitution $u(x) = y'(x)$. (iii) Newton's method. We are looking for the solution in form $y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \cdots = 1 - 2x + \frac{x^2}{2} + \cdots$.

Linear equations with constant coefficients. Superposition principle. Find the general solution of $y'' + y = xe^{-x} + \sin 2x + \cos x$. Explanation of the resonance – term $\cos x$.

Homework (optional):

Solve the initial value problem $y'' + 2y' = 3e^x$, $y(0) = 2,\ y'(0) = 1$ by 3 methods (linear equation, reducible equation, Newton's method – until the fourth order terms).

Week 4.

Lecture 7. (March 4, Tuesday).

Systems of ODEs. Theorem of existence & uniqueness.

Homogeneous linear autonomous (with constant coefficients) systems of ODEs. Eigenvalues, eigenvectors, general solution.

2-dimensional homogeneous linear systems with constant coefficients. General solution with matrix method.

Case of real eigenvalues. Typical phase portraits (stable, unstable node, saddle).

Example:$\dot{x} = -2x + y,\ \dot{y} = x - 2y,\quad a = +1$, +9, the phase portrait is a stable node and a saddle.

Another example (Homework optional) $\dot{x} = 4x - y$, $\dot{y} = -9x + 4y$, the phase portrait is an unstable node.

Lecture 8. (March 5, Wednesday).

NO LECTURE for students of the Faculty of Mechanical Engineering.

Week 5.

Lecture 9. (March 11, Tuesday).
Case of complex conjugate eigenvalues. Phase portraits (stable-unstable focus, center).
Example: $\dot{x} = 4x + y$, $\dot{y} = -9x + 4y$, the phase portrait is an unstable focus.

Homework (optional). Solve the systems of ODEs $\dot{x} = -x + ay + 1,\ \dot{y} = x - y - 1,\ a = -4, +4$
Sketch the phase portraits. Remark: the stationary point is not at the origin.
Consider the system of ODEs $\dot{x} = -x + 1,\ \dot{y} = x - y - 1$. Solve the initial value (Cauchy) problem $x(0) = 2,\ y(0) = 0$.

Second order linear equations – 2-dimensional linear systems.

Lecture 10. (March 12, Wednesday)

Sample Test 1.

Week 6.

Lecture 11. (March 18, Tuesday).

Consultation.

Nonlinear systems. Linearization. Lyapunov stability. Definition. Asymptotic stability by linearization. Example: Lyapunov stability. 3-dimensional example. Consider the Lyapunov stability of the trivial solution of system
$\dot{x} = \ln(z+1) - \sin x,\ \dot{y} = -3\,\mathrm{sh}\, y + x^2 + 2z,\ \dot{z} = -x + 1 - e^z$ by 2 methods (finding the roots of the characteristic equation, Routh.Hurwitz criterion). This problem will not be included to Test1.

Lecture 12. (March 19, Wednesday, 12:15).
Test1. R. 510.