Mathematics EP2 (BMETE90AX34)

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Detailed summary of the lectures:

Lecture 1. (February 12, 2025).

(i) Short repetition of the Lecture 25 (last semester, November 27, 2024, see in TEAMS as well). Ordinary differential equations are mathematical models for time dependent finite dimensional differentiable deterministic processes.

First order ordinary differential equations, ODEs, $y' = \frac{dy}{dx} = f(x,y)$. Definition of the solution, integral curve, trajectory. Initial value (Cauchy) problem: $y' = f(x,y). y(x_0) = y_0$. Theorem of Existence & Uniqueness for the initial value problem (without proof) with conditions $f, \frac{\partial f}{\partial y} \in C^0(D)$.

Examples. General solution. $y' = f(x)$, $y = \int f(x)dx + c$. Initial value problem $y' = y, y(0) = 1$, $y = e^x$ (guessed solution).

(ii) Special cases.

Separable equation. $\frac{dy}{dx} = g(x)H(y)$. (i) $H(y) = 0$, constant solutions. (ii) $\int \frac{dy}{H(y)} = \int g(x)\, dx + c$ (with proof).

Examples. $y' = y$, (i) $y(0) = 1, (ii)$ $y(0) = 0$. Solution (i) $y(x) = e^x$, (ii) $y(x) = 0$. $y' = 2x\cos^2 y$, (i) $y(0) = \frac{\pi}{2}, (ii)$ $y(0) = 0$. Solutions(i) $y = \frac{\pi}{2}$, (ii) $y = arctan\, x^2$.

Linear equation. $y' + P(x)y = Q(x)$. Solution: $y = ce^{\int -P(x)dx} + e^{\int -P(x)dx}\int e^{\int P(x)dx}Q(x)dx$.

Example (solving linear equations – general solution, initial value problem, sketching the integral curve(s): (i) Trivial example. $y' = y, y(0) = 1, y(0) = 0$. Solutions $y(x) = e^x, y(x) = 0$.

(ii) Non-trivial example (Homework) $xy' = 3y + x^2$, $y(1) = 0$. Solution $y = x^3 - x^2$. $0 < x$.

Literature: Thomas' Calculus Chapter 9, § 1,2.

Homework (optional): Thomas' Calculus. **9.1**.11,17, **9.2**.15, and sample tests in TEAMS and on my homepage Google: Moson Péter or direct address: http://tutor.nok.bme.hu/sandwich/general/moremoson/mo.htm .

Lecture 2. (February 19, 2025).

Orthogonal trajectories. Cartesian, polar coordinates (original family $y = cx$ $\rightarrow$ orthogonal family $x^2 + y^2 = c^2$) and reversely original family $x^2 + y^2 = c^2$ $\rightarrow$ orthogonal family $y = cx$.

Further example (homework: $y = ce^x$), find the orthogonal trajectories, sketch both families of curves.

First order autonomous differential equation $y' = f(y)$. Trajectories do not intersect each other. Analytic solution. Separable equation. (i) $f(y) = 0, y = y_1, \ldots$ constant solutions. (ii) $f(y) \neq 0, \int \frac{dy}{f(y)} = \int dx = x + c$. Geometric solution. Phase portrait (line, trajectories).

Literature: Thomas' Calculus Chapter **9, § 4, 5.**

Homework (optional): Thomas' Calculus. **9.4**.3, 7. , **9.5**.17.

Lecture 3. (February 26, 2025).

First order autonomous differential equation $y' = f(y)$. Phase portrait. Typical integral curves with inflection points ($\frac{df(y)}{dy} = 0$) . Example. $y' = y^2 - y - 2$.

ODEs of the second order. $y'' = f(x, y, y')$, $y(x_0) = y_0$, $y'(x_0) = y_0{}'$. Theorem of existence and uniqueness. Examples: gravity / $my'' = mg$, harmonic oscillator / $y'' + y = 0$.

Linear ordinary differential equations of the second order with constant coefficients. Theorem of structure.
General solution of the homogeneous equation (3 cases). Example: Solution of the harmonic oscillator equation as linear equation.
Linear ordinary differential equations of the second order with constant coefficients. General solution of the nonhomogeneous equation (if the right-hand side is a polynomial, exponential, trigonometric function).
Examples 1: Example: Solution of the harmonic oscillator equation as linear equation. Solution of the gravity, as linear equation with constant coefficients.
Example 2: Solve the initial value problem $y'' + y' = -1,\ y(0) = 1,\ y'(0) = -2$ by 2 methods (linear equation, Newton's method – until the third order terms). The idea of these methods: (i) Linear with constant coefficients. Homogeneous, characteristic equation, nonhomogeneous, resonance. (ii) Newton's method. We are looking for the solution in form $y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \cdots = 1 - 2x + \frac{x^2}{2} + \cdots$.
Find the general solution $y'' + y = e^{-x} + \sin 2x + \cos x$. Explanation of the resonance.
Homework (optional). (i) Find the general solution for the following differential equations $y'' - 2y' + 2y = 0$, $y'' + y' = 0$, $y'' - 2y' + y = 0$. (ii) Solve the initial value problem $y'' + 2y' = 3e^x$, $y(0) = 2$, $y'(0) = 1$ by 2 methods (linear equation, Newton's method – until the fourth order terms). (iii) Find the general solution of the following differential equation $y'' + y' - 2y = 1 + 2\sin x\ \cos x + 2\cosh x - 2x^2$.

Lecture 4. (March 5, 2025).

2 dimensional systems of ODEs. Theorem of existence & uniqueness.

2 dimensional homogeneous linear systems with constant coefficients. Matrices. Eigenvalues, eigenvectors. General solution, phase portraits-there systems are autonomous (in case of real eigenvalues – stable node).

Example: $\dot{x} = -2x + y,\ \dot{y} = x - 2y, \quad a = +1.$

Lecture 5. (March 12, 2025). Planned.

2 dimensional homogeneous linear systems with constant coefficients. Matrices. Eigenvalues, eigenvectors. General solution, phase portraits-these systems are autonomous (in case of real eigenvalues – saddle, complex eigenvalues – focus, center).

Example: $\dot{x} = -2x + y,\ \dot{y} = x - 2y, \quad a = +9, -1.$

Homework. (Optional)
Find the general solution, sketch the phase portrait of the system of differential equations $\dot{x} = -3x + 2y,\ \dot{y} = ax - 3y, \quad a = -2, 2, 8$.

Lecture 6. (March 19, 2025).
Sample Test 1.

Lecture 7. (March 26, 2025).
Test 1. K. 221. 8:15 a.m.