

Mathematics G3 for Mechanical Engineers (BMETE93BG03)

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Prerequisites: Mathematics G1, G2.

Program:

Classification of differential equations. Separable ordinary differential equations, linear equations with constant and variable coefficients, systems of linear differential equations with constant coefficients. Some applications of ODEs. Scalar and vector fields. Line and surface integrals. Divergence and curl, theorems of Gauss and Stokes, Green formulae. Conservative vector fields, potentials. Some applications of vector analysis. Software applications for solving some elementary problems. (4 hours/4 credits)

Literature:

Thomas' Calculus by Thomas, G.B. et al. Addison-Wesley, Several editions (ISBN0321185587)

Detailed program (Spring 2024)

Week 1.

Lecture 1. (February 12, Monday).

Ordinary differential equations are mathematical models for time dependent finite dimensional differentiable deterministic processes.

First order ordinary differential equations, ODEs, $y' = \frac{dy}{dx} = f(x, y)$. Definition of the solution, integral curve, trajectory. Initial value (Cauchy) problem: $y' = f(x, y)$. $y(x_0) = y_0$. Theorem of Existence & Uniqueness for the initial value problem (without proof) with conditions $f, \frac{\partial f}{\partial y} \in C^0(D)$.

Examples (guess the solutions). General solution. $y' = f(x)$, $y = \int f(x)dx + c$. Initial value problem $y' = y$, $y(0) = 1$, $y = e^x$. Others: Maximal interval is "smaller". $y' = 1 + y^2$, $y(0) = 0$. There is no uniqueness $y' = 3y^{\frac{2}{3}}$, $y(0) = 0$.

Special cases. Separable equations. $\frac{dy}{dx} = g(x)H(y)$. (i) $H(y) = 0$, constant solutions. (ii)

$\int \frac{dy}{H(y)} = \int g(x)dx + c$ (with proof in case of initial value problem).

Example. $y' = 2x \cos^2 y$, (i) $y(0) = \frac{\pi}{2}$, (ii) $y(0) = 0$. Solutions (i) $y = \frac{\pi}{2}$, (ii) $y = \arctan x^2$.

Lecture 2. (February 13, 2024, Tuesday).

Another special case: Linear equation. $y' + P(x)y = Q(x)$. Solution:

$y = ce^{\int -P(x)dx} + e^{\int -P(x)dx} \int e^{\int P(x)dx} Q(x)dx$ (proof by the variation of constants).

Example (solving linear equations – general solution, initial value problem, sketching the integral curve(s):

$y' = 1 + y^2, y(0) = 0, y = \tan x$ (separable equation).

$xy' = 3y + x^2, y(1) = 0$. Solution $y = x^3 - x^2, 0 < x$.

$y' \cos x + y \sin x - 1 = 0, (i) y(0) = 0, (ii) y(0) = 1$. Solutions (i) $y = \sin x$, (ii) $y = \cos x + \sin x, -\frac{\pi}{2} < x < \frac{\pi}{2}$.

Literature: Thomas' Calculus Chapter 9, § 1,2,5.

Homework (optional): Thomas' Calculus. **9.1.11,17, 9.2.15**.

Remark: $y' + 2y = 3, y(0) = 1$ is both separable and linear equation.

Week 2.

Lecture 3. (February 19, Monday).

Orthogonal trajectories. Cartesian, polar coordinates ($y = cx, x^2 + y^2 = c^2$). Further example – homework. Orthogonal trajectories $y = \ln(cx), c > 0$.

First order autonomous differential equation $y' = f(y)$. Phase portrait (line, trajectories).

Sketching the graph of some typical integral curves (with inflection points).

Lecture 4. (February 20, Tuesday).

First order autonomous differential equation $y' = f(y)$. Phase portrait (line, trajectories).

Sketching the graph of some typical integral curves (with inflection points). Examples.

$y' = y, y' = y^2 - 3y + 2, y' = \sin y$.

Approximate solution of ODEs. Newton's, Euler's methods. Example: $y' = y, y(0) = 1$. Newton's

method leads to the solution $y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. Approximate value of e . Newton: $e =$

$\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \dots$, Euler: $\left(1 + \frac{1}{n}\right)^n \xrightarrow{n \rightarrow \infty} e$.

Literature: Thomas' Calculus Chapter 9, § 3, 4, 5.

Week 3.

Lecture 5. (February 27, Tuesday).

ODEs of the second order. $y'' = f(x, y, y'), y(x_0) = y_0, y'(x_0) = y_0'$. Theorem of existence and uniqueness.

Reducible to 1st order equations 2nd order equations: $y'' = f(x, y), y'' = f(y, y')$.

Examples (solution by 2 methods – elementary ideas, later linear equations with constant coefficients): $my'' = mg - \text{gravity}, y'' + y = 0$ – harmonic oscillator.

Linear second order equations. Theorem of structure (general solution of non-homogeneous equation = general solution of homogeneous equation + particular solution of non-homogeneous equation).

Linear equations with constant coefficients. General solution of homogeneous equation (3 cases). Particular solution of nonhomogeneous linear equations with constant coefficients in special case of polynomial, exponential trigonometric right-hand sides.

Lecture 6. (February 28, Wednesday).

Solve the initial value problem $y'' + y' = -1$, $y(0) = 1$, $y'(0) = -2$ by 3 methods (linear equation, reducible equation, Newton's method – until the third order terms). The idea of these methods: (i) Linear with constant coefficients. Homogeneous, characteristic equation, nonhomogeneous, resonance. (ii) Substitution $u(x) = y'(x)$. (iii) Newton's method. We are looking for the solution in form $y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \dots = 1 - 2x + \frac{x^2}{2} + \dots$.

Linear equations with constant coefficients. Superposition principle. Find the general solution of $y'' + y = xe^{-x} + \sin 2x + \cos x$.

Homework (optional):

Solve the initial value problem $y'' + 2y' = 3e^x$, $y(0) = 2$, $y'(0) = 1$ by 3 methods (linear equation, reducible equation, Newton's method – until the fourth order terms).

Week 4.

Lecture 7. (March 4, Monday).

Solution of examples related to 1st, 2nd order ODEs.

Lecture 8. (March 5, Tuesday).

Systems of ODEs. Theorem of existence & uniqueness.

Homogeneous linear autonomous (with constant coefficients) systems of ODEs. Eigenvalues, eigenvectors, general solution.

2-dimensional homogeneous linear systems with constant coefficients. General solution with matrix method.

Case of real eigenvalues. Typical phase portraits (stable, unstable node, saddle).

Example (solved at the lecture): $\dot{x} = 4x - y$, $\dot{y} = -9x + 4y$, the phase portrait is an unstable node. Another example (Homework optional) $\dot{x} = 4x - 4y$, $\dot{y} = -9x + 4y$, the phase portrait is a saddle.

Week 5.

Lecture 9. (March 11, Monday).

Case of complex conjugate eigenvalues. Phase portraits (stable-unstable focus, center).

Example: $\dot{x} = 4x + y$, $\dot{y} = -9x + 4y$, the phase portrait is an unstable focus.

Consider the system of ODEs $\dot{x} = -x + 1$, $\dot{y} = x - y - 1$. Solve the initial value (Cauchy) problem $x(0) = 2$, $y(0) = 0$

Homework (optional). Solve the systems of ODEs $\dot{x} = -x + ay + 1$, $\dot{y} = x - y - 1$, $a = -4, +4$. Sketch the phase portraits. Remark: the stationary point is not at the origin.

Lecture 10. (March 12, Tuesday)

Sample Test 1.

Reduction of 2-dimensional systems to 2nd order equations.

Week 6.

Lecture 11. (March 18, Monday).

Consultation before Test1.

Harmonic oscillator (as second order equation and 2-dimensional system):

$\ddot{x} + x = 0 \Leftrightarrow \dot{x} = y, \dot{y} = -x$. Remark related to the swing, nonlinear model $\ddot{x} + \sin x = 0 \Leftrightarrow \dot{x} = y, \dot{y} = -\sin x$, its linearization leads to $\ddot{x} + x = 0 \Leftrightarrow \dot{x} = y, \dot{y} = -x$.

Phase portrait of both, linear, nonlinear systems.

Lecture 12. (March 19, Tuesday).

Test 1.

Week 7.

Lecture 13. (March 25, Monday).

Scalar fields (invariant, Cartesian, cylindrica, spherical coordinate form). Derivation (gradient, Nabla operator). $u = u(\vec{r}) = u(x, y, z) = u(r, \varphi, h) = u(R, \theta, \varphi)$. Example: $u(\vec{r}) = \vec{r}^2 = x^2 + y^2 + z^2 = r^2 + h^2 = R^2$ Vector fields $\vec{v}(\vec{r}) = v^1(x, y, z)\vec{i} + v^2(x, y, z)\vec{j} + v^3(x, y, z)\vec{k}$. Nabla operator:

$\nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$. Derivation. Gradient (vector field): $\nabla u = \vec{i} \frac{\partial u}{\partial x} + \vec{j} \frac{\partial u}{\partial y} + \vec{k} \frac{\partial u}{\partial z}$,

derivative tensor (matrix), divergence (scalar field): $\nabla \cdot \vec{v} = \frac{\partial v^1}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial v^3}{\partial z}$.

Example: $\text{gradu}(\vec{r}) = \text{grad} \vec{r}^2 = \nabla(x^2 + y^2 + z^2) = 2x\vec{i} + 2y\vec{j} + 2z\vec{k} = 2\vec{r}$.

Remarks Test 1. Viewing, final results.

Lecture 14. (March 26, Tuesday).

Rotation (vector field): $\nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v^1 & v^2 & v^3 \end{vmatrix}.$

Second order NABLA operators. Laplace operator. $\Delta u = \text{div} \text{gradu}(\vec{r}) = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$. $\text{rot} \text{gradu}(\vec{r}) = \nabla \times \nabla u(\vec{r}) = \vec{0}$, $\text{div} \text{rot} \vec{v}(\vec{r}) = \nabla \cdot (\nabla \times \vec{v}(\vec{r})) = \vec{0}$.

Rules of derivation (partly by proofs). Example: $\text{div}(\vec{r}^2 \vec{r}) = 5\vec{r}^2$.

Introduction to potential theory. A vector field is potential, if there exists a scalar field such that its gradient is equal to the vector field - $\text{gradu} = \vec{v}$. Necessary condition: $\text{rot} \vec{v}(\vec{r}) = \vec{0}$. Example (trivial): $\vec{v}(\vec{r}) = 2\vec{r}$. $u(\vec{r}) = \vec{r}^2$.

Theoretical example – gravity. $\vec{v}(\vec{r}) = \text{gradu}(\vec{r})$, $u(\vec{r}) = 1/|\vec{r}|$.

Finding the potential function. Further (non-trivial) examples.

Cartesian coordinates. Find λ if the vector fields are potential. Determine the potential functions. $\vec{v}(\vec{r}) = (2x - 2z)\vec{i} + (2y + z)\vec{j} + (y - \lambda x)\vec{k}$.