Mathematics G3 for Mechanical Engineers (BMETE93BG03)

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Prerequisites: Mathematics G1, G2.

Program:

Classification of differential equations. Separable ordinary differential equations, linear equations with constant and variable coefficients, systems of linear differential equations with constant coefficients. Some applications of ODEs. Scalar and vector fields. Line and surface integrals. Divergence and curl, theorems of Gauss and Stokes, Green formulae. Conservative vector fields, potentials. Some applications of vector analysis. Software applications for solving some elementary problems. (4 hours/4 credits)

Literature:

Thomas' Calculus by Thomas, G.B. et al. Addison-Wesley, Several editions (ISBN0321185587)

Detailed program (Spring 2024)

Week 1.

Lecture 1. (February 12, Monday).

Ordinary differential equations are mathematical models for time dependent finite dimensional differentiable deterministic processes.

First order ordinary differential equations, ODEs, $y' = \frac{dy}{dx} = f(x, y)$. Definition of the solution, integral curve, trajectory. Initial value (Cauchy) problem: $y' = f(x, y). y(x_0) = y_0$. Theorem of Existence & Uniqueness for the initial value problem (without proof) with conditions $f, \frac{\partial f}{\partial y} \in C^0(D)$.

Examples (guess the solutions). General solution. $y' = f(x)$, $y = \int f(x)dx + c$. Initial value problem $y' = y, y(0) = 1$, $y = e^x$. Others: Maximal interval is "smaller".$y' = 1 + y^2, y(0) = 0$. There is no uniqueness $y' = 3y^{\frac{2}{3}}, y(0) = 0$.

Special cases. Separable equations. $\frac{dy}{dx} = g(x)H(y)$. (i) $H(y) = 0$, constant solutions. (ii)

$\int \frac{dy}{H(y)} = \int g(x)dx + c$ (with proof in case of initial value problem).

Example. $y' = 2x \cos^2 y$, $(i)\ y(0) = \frac{\pi}{2}, (ii)\ y(0) = 0$. Solutions$(i)\ y = \frac{\pi}{2}$, $(ii)\ y = \arctan x^2$.

Lecture 2. (February 13, 2024, Tuesday).

Another special case: Linear equation. $y' + P(x)y = Q(x)$. Solution:

$y = ce^{\int -P(x)dx} + e^{\int -P(x)dx}\int e^{\int P(x)dx}Q(x)dx$ (proof by the variation of constants).

Example (solving linear equations – general solution, initial value problem, sketching the integral curve(s):

.$y' = 1 + y^2, y(0) = 0$, $y = \tan x$ (separable equation).
$xy' = 3y + x^2,\ \ y(1) = 0$. Solution $y = x^3 - x^2$. $0 < x$.
$y' \cos x + y \sin x - 1 = 0,\ (i)\ y(0) = 0,\ (ii)\ y(0) = 1$). Solutions (i) $y = \sin x$, (ii) $y = \cos x + \sin x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Literature: Thomas' Calculus Chapter 9, § 1,2,5.

Homework (optional): Thomas' Calculus. **9.1**.11,17, **9.2**.15.

Remark: $y' + 2y = 3, y(0) = 1$ is both separable and linear equation.

Week 2.

Lecture 3. (February19, Monday).

Orthogonal trajectories. Cartesian, polar coordinates ($y = cx,\ x^2 + y^2 = c^2$). Further example – homework. Orthogonal trajectories $y = \ln(cx)$, $c > 0$.
First order autonomous differential equation $y' = f(y)$. Phase portrait (line, trajectories).
Sketching the graph of some typical integral curves (with inflection points).

Lecture 4. (February 20, Tuesday).

First order autonomous differential equation $y' = f(y)$. Phase portrait (line, trajectories).
Sketching the graph of some typical integral curves (with inflection points). Examples.
$y' = y$, $y' = y^2 - 3y + 2$, $y' = \sin y$.

Approximate solution of ODEs. Newton's, Euler's methods. Example: $y' = y,\ y(0) = 1$. Newton's method leads to the solution $y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. Approximate value of e . Newton: $e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \ldots$, Euler: $\left(1 + \frac{1}{n}\right)^n \xrightarrow[n \to \infty]{} e$.

Literature: Thomas' Calculus Chapter **9, § 3, 4, 5**.

Week 3.

Lecture 5. (February27, Tuesday).

ODEs of the second order. $y'' = f(x, y, y')$, $y(x_0) = y_0$, $y'(x_0) = y_0{}'$. Theorem of existence and uniqueness.
Reducible to 1st order equations 2nd order equations: y"=f(x,y'), y"=f(y,y').

Examples (solution by 2 methods – elementary ideas, later linear equations with constant coefficients): $my'' = mg$ – gravity, $y'' + y = 0$ – harmonic oscillator.

Linear second order equations. Theorem of structure (general solution of non-homogeneous equation = general solution of homogeneous equation + particular solution of non-homogeneous equation.

Linear equations with constant coefficients. General solution of homogeneous equation (3 cases). Particular solution of nonhomogeneous linear equations with constant coefficients.in special case of polynomial, exponential trigonometric right-hand sides.

Lecture 6. (February 28, Wednesday).

Solve the initial value problem $y'' + y' = -1,\ y(0) = 1,\ y'(0) = -2$ by 3 methods (linear equation, reducible equation, Newton's method – until the third order terms). The idea of these methods: (i) Linear with constant coefficients. Homogeneous, characteristic equation, nonhomogeneous, resonance. (ii) Substitution $u(x) = y'(x)$. (iii) Newton's method. We are looking for the solution in form $y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \cdots = 1 - 2x + \frac{x^2}{2} + \cdots$.

Linear equations with constant coefficients. Superposition principle. Find the general solution of $y'' + y = xe^{-x} + sin\, 2\, x + cos\, x$.

Homework (optional):
Solve the initial value problem $y'' + 2y' = 3e^x$, $y(0) = 2,\ y'(0) = 1$ by 3 methods (linear equation, reducible equation, Newton's method – until the fourth order terms).

Week 4.

Lecture 7. (March 4, Monday).

Solution of examples related to 1st, 2nd order ODEs.

Lecture 8. (March 5, Tuesday).

Systems of ODEs. Theorem of existence & uniqueness.

Homogeneous linear autonomous (with constant coefficients) systems of ODEs. Eigenvalues, eigenvectors, general solution.

2-dimensional homogeneous linear systems with constant coefficients. General solution with matrix method.

Case of real eigenvalues. Typical phase portraits (stable, unstable node, saddle).
Example (solved at the lecture): $\dot{x} = 4x - y$, $\dot{y} = -9x + 4y$, the phase portrait is an unstable node. Another example (Homework optional) $\dot{x} = 4x - 4y$, $\dot{y} = -9x + 4y$, the phase portrait is a saddle.

Week 5.

Lecture 9. (March 11, Monday).

Case of complex conjugate eigenvalues. Phase portraits (stable-unstable focus, center).
Example: $\dot{x} = 4x + y$, $\dot{y} = -9x + 4y$, the phase portrait is an unstable focus.

Consider the system of ODEs $\dot{x} = -x + 1,\ \dot{y} = x - y - 1$. Solve the initial value (Cauchy) problem $x(0) = 2,\ y(0) = 0$

Homework (optional). Solve the systems of ODEs $\dot{x} = -x + ay + 1,\ \dot{y} = x - y - 1,\ a = -4, +4$ Sketch the phase portraits. Remark: the stationary point is not at the origin.

Lecture 10. (March 12, Tuesday)

Sample Test 1.

Reduction of 2-dimensional systems to 2nd order equations.

Week 6.

Lecture 11. (March 18, Monday).

Consultation before Test1.

Harmonic oscillator (as second order equation and 2-dimensional system): $\ddot{x} + x = 0 \quad \Leftrightarrow \quad \dot{x} = y,\ \dot{y} = -x$. Remark related to the swing, nonlinear model $\ddot{x} + \sin x = 0 \quad \Leftrightarrow \quad \dot{x} = y,\ \dot{y} = -\sin x$, its linearization leads to $\ddot{x} + x = 0 \quad \Leftrightarrow \quad \dot{x} = y,\ \dot{y} = -x$. Phase portrait of both, linear, nonlinear systems.

Lecture 12. (March 19, Tuesday).

Test 1.

Week 7.

Lecture 13. (March 25, Monday).

Scalar fields (invariant, Cartesian, cylindrica, spherical coordinate form). Derivation (gradient, Nabla operator). $u = u(\bar{r}) = u(x, y, z) = u(r, \varphi, h) = u(R, \theta, \varphi)$. Example: $u(\bar{r}) = \bar{r}^2 = x^2 + y^2 + z^2 = r^2 + h^2 = R^2$ Vector fields $\bar{v}(\bar{r}) = v^1(x, y, z)\bar{\imath} + v^2(x, y, z)\bar{\jmath} + v^3(x, y, z)\bar{k}$. Nabla operator: $\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$. Derivation. Gradient (vector field): $\nabla u = \bar{i}\frac{\partial u}{\partial x} + \bar{j}\frac{\partial u}{\partial y} + \bar{k}\frac{\partial u}{\partial z}$), derivative tensor (matrix), divergence (scalar field): $\nabla \cdot \bar{v} = \frac{\partial v^1}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial v^3}{\partial z}$.

Example: $gradu(\bar{r}) = grad\bar{r}^2 = \nabla(x^2 + y^2 + z^2) = 2x\vec{\imath} + 2\vec{y}\vec{\jmath} + 2z\vec{k} = 2\vec{r}$.

Remarks Test 1. Viewing, final results.

Lecture 14. (March 26, Tuesday).

Rotation (vector field): $\nabla\times\bar{v}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ v^1 & v^2 & v^3\end{vmatrix}$.

Second order NABLA operators. Laplace operator. $\Delta u = divgradu(\bar{r}) = \nabla^2 u = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$. $rotgradu(\bar{r}) = \nabla\times\nabla u(\vec{r}) = \vec{0}$, $div\,rot\,\vec{v}(\bar{r}) = \nabla\cdot\left(\nabla\times\vec{v}(\vec{r})\right) = \vec{0}$.

Rules of derivation (partly by proofs). Example: $div\,(\bar{r}^2\vec{r}\,) = 5\bar{r}^2$.

Introduction to potential theory. A vector field is potential, if there exists a scalar field such that its gradient is equal to the vector field - $gradu = \bar{v}$. Necessary condition: $rot\bar{v}(\bar{r}) = \bar{0}$. Example (trivial): $\bar{v}(\bar{r}) = 2\vec{r}$. $u(\bar{r}) = \bar{r}^2$.

Theoretical example – gravity. $\vec{v}(\vec{r}) = gradu(\vec{r}),\ u(\vec{r}) = 1/|\vec{r}|$.

Finding the potential function. Further (non-trivial) examples.

Cartesian coordinates. Find λ if the vector fields are potential. Determine the potential functions. $\vec{v}(\vec{r})=(2x-2z)\vec{i}+(2y+z)\vec{j}+(y-\lambda x)\vec{k}$.

Week 8.

Lecture 15. (Monday, April 8, 2024).

Finding the potential function. Further (non-trivial) examples.

Vector form. Find λ if the vector fields are potential. Determine the potential functions. $\vec{v}(\vec{r})=\vec{r}^{\,2}\,\vec{k}+\lambda(\vec{k}\cdot\vec{r})\vec{r},\ \vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$.

Definition of a curve (dimension 3, γ: $\bar{r}(t) = x(t)\bar{\imath}+y(t)\bar{\jmath}+z(t)\bar{k}, \left|\dot{\bar{r}}(t)\right| \neq 0,\ \alpha\leq t\leq \omega$)). Example: Straight line.

Definition of line (curvilinear) integral $\oint_\gamma \vec{v}(\vec{r})d\vec{r} = \int_\alpha^\omega \vec{v}\left(\vec{r}(t)\right)\dot{\vec{r}}(t)dt$. Work. Line integral over straight line, circle.

Lecture 16. (Tuesday, April 9).

Retake 1.
The students whose mark of Test 1 was less than 15% or were absent must take part for the signature (there will be another, last Retake on May 28, 2024 registration in NEPTUN for this one will be necessary). If a student wants to improve his /her mark, he /she may take part at this retake (April 9). Please understand if you take part then the new mark will overwrite the old one. There is NO improvement on May 28.
There will be no lecture for other students.

Week 9.

Lecture 17. (Monday, April 15, 2024).
Viewing of Retake 1.
Arc length of a curve. $l = \int_\alpha^\varpi |\dot{\bar{r}}(t)|\,dt$. Example: helix ($\bar{r}(t) = a\cos t\,\bar{\imath} + a\sin t\,\bar{\jmath} + bt\bar{k},\ 0 \le t \le 2\pi$).

Potential theory. Theorem. Let $\bar{v} \in C^1(D), D \subset R^3$ a simply connected domain. The following 3 statements are equivalent:

1. $\bar{v}$ is potential (there exists a scalar field u such, that $gradu = \nabla u = \bar{v}$).
2. $\bar{v}$ is conservative (integral over any closed curve is 0) $\Leftrightarrow$ $\int_\gamma \bar{v}(\bar{r})d\bar{r} = u(\bar{r}(\omega)) - u(\bar{r}(\alpha))$).
3. $\bar{v}$ is rotation free $rot\bar{v}(\bar{r}) = \nabla \times \bar{v}(\bar{r}) = \bar{0}$.

Calculation of line integrals by definition and by the help of Potential theory in invariant and coordinate forms.

Theoretical example $\vec{v}(\vec{r}) = gradu(\vec{r}),\ u(\vec{r}) = 1/|\vec{r}|$.

Line integrals. Numerical exercises (in coordinates, in invariant form). Another – similar exercises were solved at the lecture.
Find the value of λ, if $\vec{v}(\vec{r}) = (2x - 3z)\vec{\imath} + 2(y + z)\vec{\jmath} + (2y - \lambda x + 3z^2)\vec{k}$ is potential. Find a potential function. For this value of λ calculate the line integral along the segment connecting the points $A(0,1,0)$ and $B(1,1,3)$ by 2 methods (i) definition of the integral, (ii) by the help of the potential theory.
Homework (optional). Thomas Chapter 16. § 2,3,8. Exercises 3. Thomas Chapter 16. § 3. Exercises 15, 21.

Lecture 18. (April 16, Tuesday).

Potential theory. Further example: Prove that the vector field is potential $\vec{v}(\vec{r}) = (\vec{j}\cdot\vec{r})(\vec{k}\cdot\vec{r})\vec{i} + (\vec{i}\cdot\vec{r})(\vec{k}\cdot\vec{r})\vec{j} + (\vec{i}\cdot\vec{r})(\vec{j}\cdot\vec{r})\vec{k},\ \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. Find $\int_{\gamma_1} \vec{v}(\vec{r})d\vec{r}$ if

$\gamma_1:\quad \bar{r}(t) = 4\cos t\,\bar{i} + 4\sin t\,\bar{j} + 3t\,\bar{k},\ 0 \le t \le \dfrac{\pi}{4}$.

Definition of a simply connected region. Theoretical counterexample $u(\vec{r}) = arctan\frac{y}{x}$, $\vec{v}(\vec{r}) = gradu(\vec{r})$, integration over the unit circle.

Week 10.

Lecture 19. (April 22, Monday)
Definition of a surface $F: \bar{r} = \bar{r}(u,v), \bar{r} \in C^1(T), \bar{r}_u \times \bar{r}_v \ne \vec{0}$. Surface area $\mu F = \iint_T |\vec{r}_u \times \vec{r}_v|\,dudv$.
Definition of the surface integral. $\iint_F \bar{v}(\bar{r})\cdot d\bar{F} = \iint_T \bar{v}(\bar{r}(u,v)\cdot(\bar{r}_u \times \bar{r}_v)dudv$. Meaning: flux.

Stokes' general theorem. Its special cases.

(i) Potential theory: $\int_\gamma \vec{v}(\vec{r})d\vec{r} = \int_\gamma \nabla u = \int_{\partial\gamma} u = u(\vec{r}(\omega)) - u(\vec{r}(\alpha))$

(ii) Stokes special theorem. $\oint_\gamma \vec{v}(\vec{r})d\vec{r} = \int_{\partial F} \vec{v} = \int_F \nabla \times \vec{v} = \iint_F rot\vec{v}(\vec{r})d\vec{F}$.

(iii) Gauss-Ostrogradsky theorem: $\oiint_F \vec{v}(\vec{r})\, d\vec{F} = \int_{\partial V} \vec{v} = \int_V \nabla \cdot \vec{v} = \iiint_V div\vec{v}dV$.

Exercise. Calculate $\oiint_F \vec{v}(\vec{r})d\vec{F}$ if $\vec{v}(\vec{r}) = \bar{r}$, F: $x^2 + y^2 + z^2 = 1$ unit sphere, normal points outward. By 3 methods (i) elementary, from the definition of surface integral, (ii) definition of surface integral, (iii) Gauss-Ostrogradsky theorem, divergence part.

Lecture 20 (April 23, Tuesday).

Gauss-Ostrogradsky theorem. $\oiint_F \vec{v}(\vec{r})\, d\vec{F} = \iiint_V div\vec{v}dV$. Example: $\vec{v}(\vec{r}) = (2x - z)\vec{\imath} + (y + z)\vec{\jmath} + z^2\vec{k}$. Calculate $\oiint_{Fi} \vec{v}(\vec{r})d\vec{F}$, $i = 1, 2, 3$ by the Gauss-Ostrogradsky theorem, if the closed surface is the boundary of the domain given by the conditions (normal points outward): F_1: $x^2 + y^2 + z^2 \le 1$, $z \ge 0$, F_2 : $\sqrt{x^2 + y^2} \le z \le 1 + \sqrt{1 - x^2 - y^2}$, F_3 : $0 \le x \le 1$, $x^2 \le y \le \sqrt{x}$, $0 \le z \le 1$.

Homework (optional). Thomas. Chapter 16. §. 5., exercises 1, 3, 21.