Mathematics EP2 (BMETE90AX34)

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Detailed summary of the lectures:

Lecture 1. (February 14, 2024).

Ordinary differential equations are mathematical models for time dependent finite dimensional differentiable deterministic processes.

First order ordinary differential equations, ODEs, $y' = \frac{dy}{dx} = f(x, y)$. Definition of the solution, integral curve, trajectory. Initial value (Cauchy) problem: $y' = f(x, y). y(x_0) = y_0$. Theorem of Existence & Uniqueness for the initial value problem (without proof) with conditions $f, \frac{\partial f}{\partial y} \in C^0(D)$.

Examples. General solution. $y' = f(x)\,, y = \int f(x)dx + c$. Initial value problem $y' = y, y(0) = 1\,, y = e^x$ (guessed solution).

Special cases.

Separable equation. $\frac{dy}{dx} = g(x)H(y)$. (i) $H(y) = 0$, constant solutions. (ii) $\int \frac{dy}{H(y)} = \int g(x)\,dx + c$ (with proof).

Examples. $y' = y,\ (i)\ y(0) = 1, (ii)\ y(0) = 0$. Solution (i) $y(x) = e^x$, (ii) $y(x) = 0$.

Linear equation. $y' + P(x)y = Q(x)$. Solution: $y = ce^{\int -P(x)dx} + e^{\int -P(x)dx} \int e^{\int P(x)dx} Q(x)dx$.

Example (solving linear equations – general solution, initial value problem, sketching the integral curve(s): (i) $y' = y,\ y(0) = 1, y(0) = 0$. Solutions $y(x) = e^x,\ y(x) = 0$ . . (ii) $xy' = 3y + x^2,\ \ y(1) = 0$. Solution $y = x^3 - x^2$. $0 < x$.

Literature: Thomas' Calculus Chapter 9, § 1,2.

Homework (optional): Thomas' Calculus. **9.1**.11,17, **9.2**.15, and sample tests in TEAMS and on my homepage Google: Moson Péter or direct address: http://tutor.nok.bme.hu/sandwich/general/moremoson/mo.htm .

Lecture 2. (February 21, 2024).

Orthogonal trajectories. Cartesian, polar coordinates (original family $y = cx\ \rightarrow$ orthogonal family $x^2 + y^2 = c^2$) and reversely original family $x^2 + y^2 = c^2\ \rightarrow$ orthogonal family $y = cx$.

Further example, homework: $y = ce^x$, find the orthogonal trajectories, sketch both families of curves.

First order autonomous differential equation $y' = f(y)$. Trajectories do not intersect each other. Analytic solution. Separable equation. (i) $f(y) = 0, y = y_1, \ldots$ constant solutions. (ii) $f(y) \neq 0, \int \frac{dy}{f(y)} = \int dx = x + c$. Geometric solution. Phase portrait (line, trajectories).

Literature: Thomas' Calculus Chapter **9, § 4, 5.**

Homework (optional): Thomas' Calculus. **9.4**.3, 7. , **9.5**.17.

Lecture 3. (February 28, 2024).

First order autonomous differential equation $y' = f(y)$. Phase portrait. Typical integral curves with inflection points ($\frac{df(y)}{dy} = 0$) . Example. $y' = y^2 - y - 2$.

ODEs of the second order. $y'' = f(x, y, y')$, $y(x_0) = y_0$, $y'(x_0) = y_0{}'$. Theorem of existence and uniqueness. Examples: gravity / $my'' = mg$, harmonic oscillator / $y'' + y = 0$.

Linear ordinary differential equations of the second order with constant coefficients. Theorem of structure. General solution of the homogeneous equation (3 cases).
Example: Solution of the harmonic oscillator, as linear equation.
Homework (optional). Find the general solution for the following differential equations $y'' - 2y' + 2y = 0$, $y'' + y' = 0$, $y'' - 2y' + y = 0$.

Lecture 4. (March 6, 2024).
Linear ordinary differential equations of the second order with constant coefficients. General solution of the nonhomogeneous equation (if the right-hand side is a polynomial, exponential, trigonometric function).
Example: Solution of the gravity, as linear equation with constant coefficients.
FSolve the initial value problem $y'' + y' = -1,\ y(0) = 1,\ y'(0) = -2$ by 2 methods (linear equation, Newton's method – until the third order terms). The idea of these methods: (i) Linear with constant coefficients. Homogeneous, characteristic equation, nonhomogeneous, resonance. (ii) Newton's method. We are looking for the solution in form $y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \cdots = 1 - 2x + \frac{x^2}{2} + \cdots$.
Homework (optional). Solve the initial value problem $y'' + 2y' = 3e^x$, $y(0) = 2,\ y'(0) = 1$ by 2 methods (linear equation, Newton's method – until the fourth order terms).

Lecture 5. (March 13, 2024).
Find the general solution $y'' + y = e^{-x} + \sin 2x + \cos x$. Explanation of the resonance.

2 dimensional systems, autonomous systems, linear systems with constant coefficients. Matrices. Eigenvalues, eigenvectors. General solution. General solution, phase portraits (in case of real eigenvalues – saddle, node.

Homework. (Optional)
Find the general solution of the following differential equation $y'' + y' - 2y = 1 + 2\sin x\ \cos x + 2\cosh x - 2x^2$.

Lecture 6. (March 20, 2024). Planned.

Sample Test 1.

General solution, phase portraits (in case of real eigenvalues – saddle, node - complex eigenvalues – focus, center).

Example: $\dot{x} = -2x + ay,\ \dot{y} = x - 2y, \quad a = -1,\ +1,\ +9.$

Equations - systems. $\ddot{x} + x = 0 \quad \Leftrightarrow \quad \dot{x} = y,\ \dot{y} = -x.$

Homework. (Optional)

Find the general solution, sketch the phase portrait of the system of differential equations $\dot{x} = -3x + 2y,\ \dot{y} = ax - 3y, \quad a = -2, 2, 8$. Remark: systems will not be included to Test1.

Lecture 7. (March 27, 2024).

Test 1. K. 221. 8:15 a.m.

Lecture 8. (April 10, 2024)

Functions of 2 real variables f:$D_f \subset R^2 \to R^1$. Graph - $\Gamma_f = \{(x,y,z) \,|\, z = f(x,y), (x,y) \in D_f\} \subset R^3$. Level lines - $f(x, y) = c, c \in R$. Partial derivatives - with respect to the variable x: (i) definition $\frac{\partial f(x,y)}{\partial x} = f_x'(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$, (ii) formally you consider the variable y as a constant and derive with respect to x, and with respect to the variable y: (i) definition $\frac{\partial f(x,y)}{\partial y} = f_y'(x,y) = \lim_{\Delta x \to 0} \frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$, (ii) formally you consider the variable x as a constant and derive with respect to y. Tangent plane at (x_0, y_0): $z - f(x_0,y_0) = f_x'(x_0,y_0)(x - x_0) + f_y'(x_0,y_0)(y - y_0)$. Examples: $f_1(x,y)=6-2x-3y$, $f_2(x,y) = \sqrt{(4-x^2-y^2)}$, $f_3(x,y) = x^2+y^2$., $f_4(x,y) = x^2-y^2$ Graphs: z= 6-2x-3y plane, $z = \sqrt{(4-x^2-y^2)}$ semi-sphere, $z = x^2 + y^2$ paraboloid of revolution, z= x^2-y^2 saddle.

Partial derivatives: $\frac{\partial f_1(x,y)}{\partial x} = -2$, $\frac{\partial f_1(x,y)}{\partial y} = -3$, $\frac{\partial f_2(x,y)}{\partial x} = \frac{-x}{\sqrt{(4-x^2-y^2)}}$, $\frac{\partial f_2(x,y)}{\partial y} = \frac{-y}{\sqrt{(4-x^2-y^2)}}$, $\frac{\partial f_3(x,y)}{\partial x} = 2x$, $\frac{\partial f_3(x,y)}{\partial y} = 2y$, $\frac{\partial f_4(x,y)}{\partial x} = 2x$, $\frac{\partial f_4(x,y)}{\partial y} = -2y$.

Tangent plane at $(x_0, y_0) = (0,0)$: z= 6-2x-3y, z = 2 , z= 0, Z=0.

Homework (optional). Thomas. Chapter 14, §1, Exercises 7, 9. Chapter 14, §3, Exercises 7, 9, 43. Chapter 14, §6, Exercises 9, 11.

Lecture 9. (April 17, 2024)

Local extremum of functions of 2 real variables. Necessary ($\frac{\partial f(x,y)}{\partial x} = \frac{\partial f(x,y)}{\partial y} = 0 \to (x*,\ y*)$) and sufficient ($\det\begin{pmatrix} f''_{xx}\,(x*,y*) & f''_{xy}\,(x*,y*) \\ f''_{yx}\,(x*,y*) & f''_{yy}\,(x*,y*) \end{pmatrix}^{>0,\ f''_{xx}>0 \text{ MIMIMUM}, f''_{xx}<0 \text{ MAXIMUM}}_{<0,\ SADDLE}$) conditions.

Examples for local extremum f(x,y)=x^2+y^2, x^2-y^2, $x^3+3xy+y^3$.

Functions of 2 real variables. Global extremum (on compact sets).

Examples: f(x,y)= x^2+y^2-2x on $x^2+y^2<=4$.

Homework (optional). Thomas. Chapter 14, §7, Exercise 25, 29, 31, 41.

Lecture 10. (April 24, 2023).

Double integral (Cartesian coordinates). Example: Volume of a pyramid calculated by 3 methods (elementary, integration). f(x,y)= 6-2x-3y. $D_x = \left\{(x,y) \middle| 0 \le x \le 3,\ 0 \le y \le -\frac{2}{3}x + 2\right\}$, $D_y = \left\{(x,y) \middle| 0 \le y \le 2,\ 0 \le x \le -\frac{3}{2}y + 3\right\}$. $\iint_D f(x,y)dxdy = 6$.

Substitution in double integrals. Polar coordinates. Volume of the semi sphere $f(x,y)= \sqrt{(4-x^2-y^2)}$. $\iint_D f(x,y)dxdy = \int_0^2 \int_0^{2\pi} \sqrt{4-r^2}\, r d\varphi dr = \frac{16\pi}{3}$.