Differential Equations 1 (BMETE93AM15**)**

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Detailed program (Spring 2024)

Lecture 1. (February 13, Tuesday)

Differential equations. History: Newton-Leibniz, Euler, Laplace, Lyapunov, Poincaré.

First order ordinary differential equations, ODEs, $y' = \frac{dy}{dx} = f(x,y)$. Definition of the solution, integral curve, trajectory. Initial value (Cauchy) problem: $y' = f(x,y).\, y(x_0) = y_0$. Theorem of Existence & Uniqueness for the initial value problem (without proof) with conditions $f, \frac{\partial f}{\partial y} \in C^0(D)$.

Examples.

(i) Initial value problems- guessed solutions. $y' = y, y(0) = 1.\, y(x) = e^x.\, y' = 1 + y^2, y(0) = 0, y(x) = \tan x$ – „smaller” maximal interval, $y' = 3y^{\frac{2}{3}}, y(0) = 0$ - no unicity.

(ii) General solutions. $y' = f(x)\,, y = \int f(x)dx + c\,, y' = 0\,, y = c$. Integral curves shifted along the *y* axis.

$y' = y\,, y(x) = ce^x \leftrightarrow y(x) = \pm e^{x+c}, y = 0$, exponential function, Integral curves shifted along the *x* axis.

Lecture 2. (February 20, Tuesday)

Separable equations. $\frac{dy}{dx} = g(x)H(y)$. (i) $H(y) = 0$ - constant solutions. (ii) $\int \frac{dy}{H(y)} = \int g(x)dx + c$ (with proof). Examples (i) $y' = -2xy^2$. $(i)\ y(0) = 1$, $(ii)\ y(0) = 0$. Solutions $(i)\ y = \frac{1}{1+x^2}$, $(ii)\ y = 0$. General solution. (ii) $y' = 2x\cos^2 y$, $(i)\ y(0) = \frac{\pi}{2}$, $(ii)\ y(0) = 0$. Solutions$(i)\ y = \frac{\pi}{2}$, $(ii)\ y = \arctan x^2$.

Preparation for the proof of existence and uniqueness. Equivalent integral equation. $y' = f(x,y).\, y(x_0) = y_0 \leftrightarrow y(x) = y_0 + \int_{x_0}^{x} f(\overline{x}, y(\overline{x}))d\overline{x}$.

Literature: Thomas' Calculus Chapter 9, § 1,2,5.

Homework (optional): Thomas' Calculus. **9.1**.11,17.

Lecture 3. (February 21, Wednesday).

Linear problems -theorem of structure. 1st order linear equation. $y' + P(x)y = Q(x)$. Solution:

$y = ce^{\int -P(x)dx} + e^{\int -P(x)dx}\int e^{\int P(x)dx}Q(x)dx$ (proof by substitution, by the variation of constants).

Examples (solving linear equations – general solution, initial value problem, sketching the integral curve(s): $xy' = 3y + x^2$, $y(1) = 0$. Solution $y = x^3 - x^2$. $0 < x$. $y'\cos x + y\sin x - 1 = 0$, $(i)\ y(0) = 0$, $(ii)\ y(0) = 1$). Solutions (i) $y = \sin x$, (ii) $y = \cos x + \sin x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Preparation for the proof of existence and uniqueness. Gronwall-Bellman-Bihari lemma: $u, v \in C^0_{[a,b]}, u(x) \geq 0, v(x) \geq 0, c \in R, c \geq 0, a \leq x_0 \leq x \leq b,\ u(x) \leq c + \int_{x_0}^{x} u(t)v(t)dt \ \rightarrow\ u(x) \leq ce^{\int_{x_0}^{x} v(t)dt}$ (with proof).

Literature: Thomas' Calculus Chapter 9, §. 2.

Homework (optional): Thomas' Calculus. **9.2**.15.

Lecture 4-5. (February 27, 28, Tuesday, Wednesday).

Remark on reducible equations (right side depending on y/x, Bernoulli equation to 1st order ODE). Examples will be later.
Orthogonal trajectories. Theory. Sketching the graph of both families.
Demonstrative example: Cartesian, polar coordinates ($y = cx$, $x^2 + y^2 = c^2$).
Example. Find the orthogonal trajectories, sketch both families of curves. (i) : $y = ce^x, -\frac{y^2}{2} = x + c$.
Further example. $y = \ln(cx)$, $c < 0$.
Isogonal trajectories.

Literature: Thomas' Calculus Chapter 9, §. 5.

Homework (optional): Thomas' Calculus. **9.5**.14, 17.

Lecture 6. (March 5, Tuesday).

Autonomous ODEs of the 1st order. Analytic solution – separable equation. 1-dimensional phase space (line). Stationary points. Phase portrait. Geometric solution – some typical integral curves (with inflection points).
Examples: $y' = y$, $y' = -2 + y + y^2, y' = y^3 - y^2,\ y' = \cos y$.

Lecture 7. (March 6, Wednesday).

ODEs of the second order. $y'' = f(x, y, y')$, $y(x_0) = y_0$, $y'(x_0) = y_0{}'$. Theorem of existence and uniqueness.
Reducible to 1st order equations of 2nd order equations (by substitution): $y'' = f(x, y')$, $u(x) = y'(x)$.): $y'' = f(y, y')$, $u(y(x)) = y'(x)$.

Examples (solution by 2 methods – elementary ideas, later linear equations with constant coefficients): $my'' = mg -$ gravity (linearized), $y'' + y = 0 -$ harmonic oscillator (e.g. linearized pendulum).

Linear second order equations. Theorem of structure (general solution of non-homogeneous equation = general solution of homogeneous equation + particular solution of non-homogeneous equation.

Linear equations with constant coefficients. General solution of homogeneous equation (3 cases). Particular solution of nonhomogeneous linear equations with constant coefficients.in special case of polynomial, exponential trigonometric right-hand sides.

Solve the initial value problem $y'' - y' = 1,\ y(0) = 0,\ y'(0) = 0$.

Lecture 8. (March 12, Tuesday).

Solve the initial value problem, sketch the graph of the solution $y'' - y' = 1,\ y(0) = 0,\ y'(0) = 0$ by 3 methods (linear equation, reducible equation, Newton's method – until the fourth order terms). The idea of these methods: (i) Linear with constant coefficients. Homogeneous, characteristic equation, nonhomogeneous, resonance. (ii) Substitution $u(x) = y'(x)$. (iii) Newton's method. We are looking for the solution in form $y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \cdots = 1 - 2x + \frac{x^2}{2} + \cdots$.

Linear equations with constant coefficients. Superposition principle. Find the general solution of $y'' + y = xe^{-x} + \cos 2x - \cos x$. Explanation of the resonance.

Lecture 9. (March 13, Wednesday).
Sample Test1.

Lecture 10. (March 19, Tuesday).
Consultation.
Approximate solution of ODEs. Newton's – Taylor series, Euler's - broken line, Picard's – iteration - methods. Example: $y' = y,\ y(0) = 1$. Newton's method leads to the solution $y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.
Approximate value of e . Newton: $e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \ldots$, Euler: $\left(1 + \frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} e$.
Approximate solution of ODEs. Further example (Newton, Euler, Picard, isoclines) $y' = x^2 + y^2 - 1,\ y(0) = 0$.

Literature: Thomas' Calculus Chapter **9, § 3, 4, 5**.

Lecture 11. (March 20. Wednesday).
Test1.

Lecture 12-13. (March 26,27. Tuesday, Wednesday).

Systems of ODEs. Theorem of existence & uniqueness.

Homogeneous linear autonomous (with constant coefficients) systems of ODEs. Eigenvalues, eigenvectors, general solution.

2-dimensional homogeneous linear systems with constant coefficients. General solution with matrix method. Case of real eigenvalues. Typical phase portraits (stable, unstable node, saddle). Case of complex conjugate eigenvalues. Phase portraits (stable-unstable focus, center).

Lecture 14-15. (April 9, 10. Tuesday, Wednesday).
Examples (solved at the lectures): $\dot{x} = 4x - y$, $\dot{y} = -9x + 4y$, the phase portrait is an unstable node. $\dot{x} = 4x - 4y$, $\dot{y} = -9x + 4y$, the phase portrait is a saddle.
Substitution (polar coordinates). Cases of stable, unstable focus, center. Example: $\dot{x} = 4x + y$, $\dot{y} = -9x + 4y$, the phase portrait is an unstable focus (matrix method, reduction to 2nd order equation).
Homework (optional). (i) Solve the systems of ODEs $\dot{x} = -x + ay + 1$, $\dot{y} = x - y - 1$, $a = -4$, $+4$ Sketch the phase portraits. Remark: the stationary point is not at the origin. (ii) Another example: Consider the system of ODEs $\dot{x} = -x + 1,\ \dot{y} = x - y - 1$. Solve the initial value (Cauchy) problem.

Nonlinear 2-dimensional autonomous systems. Local linearization – Poincaré theorem. Sketching the phase portrait of the Lotka-Volterra type predator-prey system: $\dot{x} = x(2 - x - y)$, $\dot{y} = y(-1 + x)$.

Lecture 16. (April 16, Tuesday).
Retake 1.
The students whose mark of Test 1 was less than 6 % or were absent must take part for the signature (there will be another, last Retake on May 28, 2024 registration in NEPTUN for this one will be necessary). If a student wants to improve his /her mark, he /she may take part at this retake (April 16). Please understand if you take part then the new mark will overwrite the old one. There is NO improvement on May 28.
There will be no lecture for other students.

Lecture 17. (April 17, Wednesday).
Lipschitz condition. Proof of uniqueness (in case of equations).
Continuous, differentiable dependence on initial conditions (case of systems). Continuous proved by Gronwall-Bihari lemma. Variational system (linearization of a nonlinear system) obtained under the condition of differentiability. Example:

Remark: Lectures start at 16:30.

Lecture 18. (April 23, Tuesday).

Lyapunov stability. Definition. Asymptotic stability by linearization. Example: Lyapunov stability. 3-dimensional example. Consider the Lyapunov stability of the trivial solution of system $\dot{x} = \ln(z+1) - \sin x,\ \dot{y} = -3\,\mathrm{sh}\,y + x^2 + 2z,\ \dot{z} = -x + 1 - e^z$ by 2 methods (finding the roots of the characteristic equation, Routh.Hurwitz criterion).

Lecture 19. (April 24, Wednesday).
Laplace transformation. Definition, table of Laplace transforms, rules (linear, derivatives, integral). Solution of 2nd order linear ODEs with constant coefficients, and 2 dimensional linear systems of ODEs by Laplace transformation.

Lectures 20-21 (April 30, May 1, 2024).
National Holiday. No lecture.

Lecture 22 (Tuesday, May 7, 2024).
2nd order linear ODEs with variable coefficients. General solution. Reduction to 1st order equation. Variation of constants. Eulers's equation. Linear system with variable coefficients. Wronski determinant.

Lecture 23 (Wednesday, May 8, 2024).
Sample Test 2.

Lecture 24 (Tuesday, May 14, 2024).
Test 2.

Lecture 25 (May 15, Wednesday).
Viewing of Test 2.
Differential equations in symmetric form. Exact ODEs. Example: separable equations. Multiplier method. Example: linear equation of the 1st order. Multiplier: $\mu(x)$.

Lecture 26 (Tuesday, May 21, 2024).
Retake 2. The students whose mark of Test 2 was less than 6 % or were absent must take part for the signature (there will be another, last Retake on May 28, 2024 registration in NEPTUN for this one will be necessary). If a student wants to improve his /her mark, he /she may take part at this retake (May 21). Please understand if you take part then the new mark will overwrite the old one. There is NO improvement on May 28.
There will be no lecture for other students.

Lecture 27 (May 22, Wednesday).
Viewing of Retake2. Summary of the course. Sample written exam.