Differential Equations 1 (BMETE93AM15)

Lecturer: *Dr. Peter Moson* (www.math.bme.hu, +3630138718)

Detailed program (Spring 2024)

Lecture 1. (February 13, Tuesday)
Differential equations. History: Newton-Leibniz, Euler, Laplace, Lyapunov, Poincaré.
First order ordinary differential equations, ODEs, $y' = \frac{dy}{dx} = f(x,y)$. Definition of the solution, integral curve, trajectory. Initial value (Cauchy) problem: $y' = f(x,y). y(x_0) = y_0$. Theorem of Existence & Uniqueness for the initial value problem (without proof) with conditions $f, \frac{\partial f}{\partial y} \in C^0(D)$.
Examples.
(i) Initial value problems- guessed solutions. $y' = y, y(0) = 1$. $y(x) = e^x$. $y' = 1 + y^2, y(0) = 0, y(x) = \tan x$ – „smaller" maximal interval, $y' = 3y^{\frac{2}{3}}, y(0) = 0$ - no unicity.
(ii) General solutions. $y' = f(x)$, $y = \int f(x)dx + c$, $y' = 0$, $y = c$. Integral curves shifted along the y axis.
$y' = y$, $y(x) = ce^x \leftrightarrow y(x) = \pm e^{x+c}, y = 0$, exponential function, Integral curves shifted along the x axis.

Lecture 2. (February 20, Tuesday)
Separable equations. $\frac{dy}{dx} = g(x)H(y)$. (i) $H(y) = 0$ - constant solutions. (ii) $\int \frac{dy}{H(y)} = \int g(x)dx + c$ (with proof). Examples (i) $y' = -2xy^2$. (i) $y(0) = 1$, (ii) $y(0) = 0$.
Solutions (i) $y = \frac{1}{1+x^2}$, (ii) $y = 0$. General solution. (ii) $y' = 2x\cos^2 y$, (i) $y(0) = \frac{\pi}{2}$, (ii) $y(0) = 0$. Solutions(i) $y = \frac{\pi}{2}$, (ii) $y = \arctan x^2$.

Preparation for the proof of existence and uniqueness. Equivalent integral equation. $y' = f(x,y). y(x_0) = y_0 \leftrightarrow y(x) = y_0 + \int_{x_0}^{x} f(\overline{x}, y(\overline{x}))d\overline{x}$.
Literature: Thomas' Calculus Chapter 9, § 1,2,5.
Homework (optional): Thomas' Calculus. **9.1**.11,17.

Lecture 3. (February 21, Wednesday).
Linear problems -theorem of structure. 1st order linear equation. $y' + P(x)y = Q(x)$. Solution:
$y = ce^{\int -P(x)dx} + e^{\int -P(x)dx} \int e^{\int P(x)dx} Q(x)dx$ (proof by substitution, by the variation of constants).
Examples (solving linear equations – general solution, initial value problem, sketching the integral curve(s): $xy' = 3y + x^2$, $y(1) = 0$. Solution $y = x^3 - x^2$. $0 < x$. $y'\cos x + y\sin x - 1 = 0$, (i) $y(0) = 0$, (ii) $y(0) = 1$). Solutions (i) $y = \sin x$, (ii) $y = \cos x + \sin x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Preparation for the proof of existence and uniqueness. Gronwall-Bellman-Bihari lemma:
$u, v \in C^0_{[a,b]}, u(x) \geq 0, v(x) \geq 0, c \in R, c \geq 0, a \leq x_0 \leq x \leq b, \ u(x) \leq c +$
$\int_{x_0}^{x} u(t)v(t)dt \ \rightarrow \ u(x) \leq c e^{\int_{x_0}^{x} v(t)dt}$ (with proof).

Literature: Thomas' Calculus Chapter 9, §. 2.

Homework (optional): Thomas' Calculus. **9.2**.15.

Lecture 4-5. (February 27, 28, Tuesday, Wednesday).

Remark on reducible equations (right side depending on y/x, Bernoulli equation to 1st order ODE). Examples will be later.
Orthogonal trajectories. Theory. Sketching the graph of both families.
Demonstrative example: Cartesian, polar coordinates ($y = cx, \ x^2 + y^2 = c^2$).
Example. Find the orthogonal trajectories, sketch both families of curves. (i) : $y =$
$ce^x, -\frac{y^2}{2} = x + c$.
Further example. $y = \ln(cx), \ c < 0$.
Isogonal trajectories.

Literature: Thomas' Calculus Chapter 9, §. 5.

Homework (optional): Thomas' Calculus. **9.5**.14, 17.

Lecture 6. (March 5, Tuesday).

Autonomous ODEs of the 1st order. Analytic solution – separable equation. 1-dimensional phase space (line). Stationary points. Phase portrait. Geometric solution – some typical integral curves (with inflection points).
Examples: $y' = y$, $y' = -2 + y + y^2, y' = y^3 - y^2, \ y' = \cos y$.

Lecture 7. (March 6, Wednesday).

ODEs of the second order. $y'' = f(x, y, y'), y(x_0) = y_0, y'(x_0) = y_0'$. Theorem of existence and uniqueness.
Reducible to 1st order equations of 2nd order equations (by substitution): $y'' = f(x, y'), u(x) = y'(x) .$): $y'' = f(y, y'), u(y(x)) = y'(x) .$

Examples (solution by 2 methods – elementary ideas, later linear equations with constant coefficients): $my'' = mg -$ gravity (linearized), $y'' + y = 0 -$ harmonic oscillator (e.g. linearized pendulum).

Linear second order equations. Theorem of structure (general solution of non-homogeneous equation = general solution of homogeneous equation + particular solution of non-homogeneous equation.

Linear equations with constant coefficients. General solution of homogeneous equation (3 cases). Particular solution of nonhomogeneous linear equations with constant coefficients.in special case of polynomial, exponential trigonometric right-hand sides.

Solve the initial value problem $y'' - y' = 1, \ y(0) = 0, \ y'(0) = 0$.

Lecture 8. (March 12, Tuesday).

Solve the initial value problem, sketch the graph of the solution $y'' - y' = 1$, $y(0) = 0$, $y'(0) = 0$ by 3 methods (linear equation, reducible equation, Newton's method – until the fourth order terms). The idea of these methods: (i) Linear with constant coefficients. Homogeneous, characteristic equation, nonhomogeneous, resonance. (ii) Substitution $u(x) = y'(x)$. (iii) Newton's method. We are looking for the solution in form $y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \cdots = 1 - 2x + \frac{x^2}{2} + \cdots$.

Linear equations with constant coefficients. Superposition principle. Find the general solution of $y'' + y = xe^{-x} + \cos 2x - \cos x$. Explanation of the resonance.

Lecture 9. (March 13, Wednesday).
Sample Test1.

Lecture 10. (March 19, Tuesday).
Consultation.
Approximate solution of ODEs. Newton's – Taylor series, Euler's - broken line, Picard's – iteration - methods. Example: $y' = y$, $y(0) = 1$. Newton's method leads to the solution $y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

Approximate value of e. Newton: $e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + ..,$ Euler: $\left(1 + \frac{1}{n}\right)^n \xrightarrow[n \to \infty]{} e$.
Approximate solution of ODEs. Further example (Newton, Euler, Picard, isoclines) $y' = x^2 + y^2 - 1$, $y(0) = 0$.

Literature: Thomas' Calculus Chapter **9, § 3, 4, 5**.

Lecture 11. (March 20. Wednesday).
Test1.

Lecture 12-13. (March 26,27. Tuesday, Wednesday).

Systems of ODEs. Theorem of existence & uniqueness.

Homogeneous linear autonomous (with constant coefficients) systems of ODEs. Eigenvalues, eigenvectors, general solution.

2-dimensional homogeneous linear systems with constant coefficients. General solution with matrix method. Case of real eigenvalues. Typical phase portraits (stable, unstable node, saddle). Case of complex conjugate eigenvalues. Phase portraits (stable-unstable focus, center).