Mathematics EP1. Detailed program (2024):

Week 1. Lectures 1-2. (September 4, 6, Wednesday, Friday).

Theory.

Numbers (natural, integer, rational, real, complex).

Real numbers. Axioms. Examples for commutativity, associativity, distributivity.

Complex numbers (definition, algebraic-polar-exponential forms, addition, subtraction, multiplication, division, de Moivre formula). Fundamental theorem of algebra.

Exercises.
Real, complex numbers. Proof of identities$(a+b)(a-b)=a^2-b^2$, by axioms. $(a+b)^2=a^2+2ab+b^2$, by division of a square.

Complex numbers.
Calculations with complex numbers. Consider the complex numbers: $z_1=1+i$, $z_2=2(cos\,90^0+i\,sin90^0)=2e^{i\frac{\pi}{2}}$. Find the value of the following complex numbers: (i) z_1-z_2 , (ii) $z_1\cdot z_2$ (iii) $\frac{z_2}{z_1}$, (iv) ${z_1}^2$, (v) ${z_1}^4$.

Theory.
Fundamental theorem of algebra.
Rotation in the plane by complex numbers.

Exercises.
Applications of complex numbers.

(i) Solve the equation (lecture) $z^2+2iz-5=0$. Further examples (homework) $,z^4-1=0$, $z^4+z^2-2=0$, $z^3+z-2=0, z^3-2z^2+z-2=0$.

(ii) 2 vertices of a square are $z_1=0$, $z_2=5+12i$. Find the other vertices, perimeter, area of this square.

Literature.
Thomas' Calculus. Chapters 1. §1, Chapter 12. §1-3. Appendices: F3. Real numbers, F4. Complex numbers.
Homework.
Thomas' Calculus**. F3** (Appendix). 7., 13., 23. **Chapter 12**. 3.1.

Week 2. Lecture 3-4 (September 11, Wednesday, September 13, Friday).

Theory.

Vectors in planar geometry. Addition, scalar (dot) product and its meaning (projection), definition of vector (cross) product. Calculation of the previous products in Cartesian coordinates. Geometric meaning of dot (scalar), vector (cross) product (projection – perpendicular vectors, area of parallelogram, parallel vectors).
*Quadratic 2*2 matrices.* Determinant and its meaning (area of the parallelograms formed by rows or columns, linear independence).

Exercise.
$OA = \vec{a} = 3\vec{\imath} + 4\vec{\jmath} = (3,4)$, $OB = \vec{b} = -8\vec{\imath} + 6\vec{\jmath} = (-8,6)$. Find $\vec{a} - \vec{b},\ \vec{a} \cdot \vec{b},\ \vec{a} \times \vec{b}$. What is the meaning of these expressions? How much is the perimeter, the area of the triangle OAB ?

Theory.

Vectors in spatial geometry. Addition, scalar (dot) product and its meaning (projection), definition of vector (cross) product. Calculation of the previous products in Cartesian coordinates. Geometric meaning of vector (cross) product (area of parallelogram).

*Quadratic 3*3 matrices.* Determinant and its meaning (volume of the parallelepipedons formed by rows or columns).

Exercises.
1. $OA = \vec{a} = 2\vec{\imath} + \vec{\jmath} + 2\vec{k} = (2,1,2)$, $OB = \vec{b} = 2\vec{\imath} - 4\vec{\jmath} = (2,-4,0)$. Find $\vec{a} - \vec{b},\ \vec{a} \cdot \vec{b},\ \vec{a} \times \vec{b}$. How much is the perimeter, the area of the triangle OAB ?
2. Volume of an arbitrary parallelepipedon.

Literature.
Thomas' Calculus. Chapter 12. §4. (cross product, determinants).
Homework.
Thomas' Calculus**. Chapter 12**. 4.15.

Week 3. Lectures 5-6. (September 17, Wednesday, September 19, Friday.)

Theory.

Functions of 1 real variable – **main subject of this semester** (domain, range, graph, even, odd, etc.).

Definition of the derivative of function of 1 real variable. $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x) = \frac{df(x)}{dx}$. Geometric meaning – slope of the tangent line. Equation of the tangent line $y - f(x_0) = f'(x_0)\cdot(x - x_0)$. Tangent lines. $f(x) = x^2,\ (x_0, y_0) = (1,1)\ , y = 2x - 1$.

Rules of derivation (sum, multiplication by constant, product – with proof, ratio, composed, inverse functions- with proof). Formulae (without exact conditions): $(u+v)' = u' + v'\ , (c\,u)' = c\,u'\ , (u\cdot v)' = u'v + uv'\ , \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}\ , (\,f\big(g(x)\big)\,)' = f'\big(g(x)\big)\cdot g'(x), [f^{-1}(x)]' = \frac{1}{f'(f^{-1}(x))}$.

Exercises.

Derivative by definition: $f(x) = x,\ x^2$. Derivative of some more elementary functions by definition (optional homework): $(x) = \frac{1}{x}\ , \sqrt{x}$. Result: $\boldsymbol{(x^n)' = n\,x^{n-1}}$ **.**

Functions of 1 real variable. Development of functions into power series, *Maclaurin polynomial, series* $f(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$.

Definition of the exponential function. $f'(x) = f(x) = e^x,\ f(0) = 1$. Maclaurin series $e^x = \sum_{k=0}^{\infty}\frac{1}{k!}x^k = 1 + x + \frac{x^2}{2!} + \cdots$. Its properties (e.g. $e^{a+b} = e^a e^b$). Approximate value of $e \approx 2{,}71$ ... Graph of the exponential function.

Logarithm: definition, derivative, graph $f(x) = \ln x\ , f'(x) = \frac{1}{x}$.

Exponential functions with bases $2, 10$.

Definition trigonometric functions. $\cos x = \frac{e^{ix}+e^{-ix}}{2}$, $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$, Their Maclaurin series. $\cos x = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k} = 1 - \frac{x^2}{2!} + \cdots$, $\sin x = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1} = x - \frac{x^3}{3!}\ldots$.

Tangent, inverse tangent function. Graphs, derivatives, Maclaurin series (later by integration). Geometrical series $\frac{1}{1-x} = \sum_{k=0}^{\infty}x^k = 1 + x + x^2 + \cdots\ \ |x| < 1$.

Summary. Table of derivatives. $(e^x)' = e^x\ , (lnx)' = \frac{1}{x}\ , (x^n)' = nx^{n-1}\ , (\sin x)' = \cos x\ , (\cos x)' = -\sin x\ ,\ (\sinh x)' = \cosh x\ ,\ (\cosh x)' = \sinh x\ ,\ \big(\tan x)' = 1 + (\tan x)^2 = \frac{1}{(\cos x)^2}\ , (\mathrm{arctan}x\big)' = \frac{1}{1+x^2}$, ...

Remark.

Definition of the exponential function is based on solution of ordinary differential equations. First order case: $y' = f(x,y). y(x_0) = y_0$ inital value problem has a unique solution if $f, \frac{\partial f}{\partial y} \in C^0(D)$.

Literature.
Thomas' Calculus. Chapter 3. §1.

Homework. (Optional).
Thomas' Calculus**. Chapter 3**. 1.7, 9.
Thomas' Calculus**. Chapter 3**. Exercises (at the end of the chapter) 3, 13, 27, 41, 63, 65.

Week 4. Lectures 7-8. (September 25, Wednesday, September 27, Friday).

Theory.

Geometrical series $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots \quad |x| < 1$.

Applications of derivation. Local approximation of functions in a neighborhood of $x = x_0$. Order „0" $f(x) \approx f(x_0)$. Order „1" $f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0)$. Tangent line $y - f(x_0) = f'(x_0) \cdot (x - x_0)$. Order „2" $f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0) + \frac{f''(x_0)}{2!} \cdot (x - x_0)^2$, by parabolas (in general): $y = f(x_0) + f'(x_0) \cdot (x - x_0) + \frac{f''(x_0)}{2!} \cdot (x - x_0)^2$.

Example. $f(x) = e^x = 1 + x + \frac{x^2}{2!}$.

Taylor series at $x = x_0$: $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$. Maclaurin series are Taylor series at $x_0 = 0$.

Exercises.

Maclaurin polynomial of $f(x) = (1 + x)^2 = 1 + 2x + x^2$. Taylor polynomial of $f(x) = 1 + 2x + x^2$ at $x_0 = -1$ is $(1 + x)^2$.
Calculation of limits at the origin ($lim_{x \to 0} \frac{\sin x}{x} = 1$, $lim_{x \to 0} \frac{\sin 3x}{x} = 3$, $lim_{x \to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} = \frac{1}{120}$).

Main applications of derivation. **Investigation of functions**.
Root at $x = x_0$ is $f(x_0) = 0$.

Increase ($f' \geq 0$), decrease ($f' \leq 0$). Necessary condition for local extremum of differentiable functions at $x = x_0$ ($f'(x_0) = 0$). Sufficient condition $f''(x_0) > 0$ minimum, $f''(x_0) < 0$ maximum.
Convex, concave functions. Necessary condition for inflection points of differentiable functions at $x = x_0$ ($f''(x_0) = 0$).

Exercises.

Global extremum, extremum on closed intervals. Theorems of Weierstrass. (i) Maximal area of a rectangular garden with perimeter 8. $f(x) = x(4 - x)$. (ii) Maximal volume of a box without roof $f(x) = x(4 - 2x)^2$ (solved at the lecture).

Homework (Optional).
Thomas' Calculus**. Chapter 3**. Exercises (at the end of the chapter) 27, 41, 69, 71, 97, 99. **Chapter 5**. §3. Ex. 11., §5. Ex. 1.

Week 5. Lectures 9-10. (October 2, Wednesday – NO LECTURE – Dean's Holiday, October 4, Friday).

Definition trigonometric (repetition), hyperbolic (new) functions. $\cos x = \frac{e^{ix}+e^{-ix}}{2}$, $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$, $\cosh x = \frac{e^{x}+e^{-x}}{2}$, $\sinh x = \frac{e^{x}-e^{-x}}{2}$. Their Maclaurin series. $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \cdots$, $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \cdots$, $\cosh x = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} = 1 + \frac{x^2}{2!} + \cdots$, $\sinh x = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1} = x + \frac{x^3}{3!} +$
Derivatives, graphs of these functions. Parametrization of ellipses, hyperbolas.

Asymptotes (horizontal, vertical). Asymptotes (oblique). Sketching the graphs of functions. Example: $f(x) = e^x + x - 1$.

Taylor series at $x = x_0$: $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$. Maclaurin series are Taylor series at $x_0 = 0$.
Demonstrative example: Find the 1st, 2nd order approximation of $f(1.1) = 1{,}331$ if $f(x) = x^3$ by its 0, 1st, 2nd, 3rd order Taylor polynomials calculated at $x_0 = 1$.

Week 6.

Lecture 11. (October 9, Wednesday).
Sample Test 1.

Lecture 12. (October 11, Friday).
Test 1. Room K. 221. Start 8:15.

Week 7.

Lectures 13-14. Design Week.

Week 8.

Lecture 15. (October 23, Wednesday).

NO lecture. National holiday.

Lecture 16. (October 25, Friday).

Indefinite integral (inverse operation of the derivation). Function F is called the indefinite integral (primitive function, antiderivative) of f , if $F'(x) = f(x)$. Notation: $F(x) = \int f(x)dx$.

Calculation of integrals by elementary method – table of integrals. E.g. $(e^x)' = e^x$ gives $\int e^x dx = e^x + c$, or slightly changed $\left(x^n\right)' = nx^{n-1}$ leads to $\int x^n dx = \frac{x^{n+1}}{n+1}$, $n \neq -1$. See Table of integrals. Thomas, Chapter 8.

Derivation rules: $(u+v)' = u' + v'$, $(c\,u)' = c\,u'$, $(u \cdot v)' = u'v + uv'$, $(\,f\big(g(x)\big)\,)' = f'\big(g(x)\big) \cdot g'(x)$ lead to integration rules. Linearity, by parts - $\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$, by substitution (today only special cases $\int \frac{f'(x)}{f(x)}dx = \ln f(x) + c$, $f(x) > 0$, $\int f^n(x)f'(x)dx = \frac{f^{n+1}(x)}{n+1} + c$, $n \neq -1$).

Calculation of integrals by methods presented: $f(x) = \tan x, \ln x,$
$\arctan x$, $(2x+1)^2$.
Development of further elementary functions into Maclaurin series: $\arctan x$, $\ln(1+x)$, $\arcsin x$. Calculation the values of $\ln 2$, π .

IMPORTANT REMARK. The derivative of an elementary function always is an elementary function, the indefinite integral of an elementary function in general is NOT an elementary function (a counterexample is e.g., e^{-x^2}).

Week 9.

Lecture 17. (October 30, Wednesday). Retake 1 in K. 364. 12:15.

The students whose mark of Test 1 was less than 6%, or were absent must take part for the signature (there will be another, last Retake on December 10, registration in NEPTUN for this one will be necessary). If a student wants to improve his /her mark, he /she may take part at this retake (registration at the lecture is necessary). Please understand if you take part then the new mark will overwrite the old one. There is NO improvement on December 10.
There will be no lecture for other students.

Lecture 18. (November 1, Friday)

National Holyday. No lecture.

Week 10.

Lecture 19. (November 6, Wednesday)

Definite integral. Definition of Riemann integral, its meaning (area). Connection between indefinite and definite integral, NEWTON-LEIBNIZ formula: $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$, $F'(x) = f(x)$.
Applications of definite integration. Main meaning – area. Example: area of a triangle ($\int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \left(\frac{1^2}{2} - \frac{0^2}{2}\right) = \frac{1}{2}$.

IMPORTANT REMARK. A general planar set does NOT have an area = a function in general is NOT integrable (even if we introduce other integrals, e.g., Lebesgue). Counterexample (in case of Riemann integral) – Dirichlet function.

Examples for non-differentiable (at $x = 0$) functions $f(x) = sign\ (x), f(x) = |x| = \sqrt{x^2}$. Example of non-integrable (by Riemann) Dirichlet function $d(x) = \begin{cases} 1\ if\ x - rational\ number \\ 0\ if\ x\ irrational\ number \end{cases}$.

Viewing of Test1, Retake1.

Lecture 20. (November 8, Friday)

Definite integral variants of the formulae of integration, especially by parts - $\int_a^b u'(x)v(x)dx = [u(x)v(x)]_a^b - \int_a^b u(x)v'(x)dx$, and by substitution - $\int_a^b f(x)dx = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(t))\varphi'(t)dt$.

Examples. $\int_0^{\frac{\pi}{2}} e^x \sin x\, dx$, area of the circle.

Homework – applications of integration (optional): Thomas 4.4.35, 37. 4.6.11,23. 6.1.45/and 6.4.13, 15.

Week 11.

Lecture 21. (November 13, Wednesday)

Application of integration. Area, arc length $l = \int_a^b \sqrt{1 + f'(x)^2}\, dx$ of a circle.

Mass of a homogeneous planar plate ($m = \int_a^b f(x)\rho dx$), center of mass (gravity): $x_c = \frac{\int_a^b x f(x)\rho dx}{m}$, $y_c = \frac{\frac{1}{2}\int_a^b f^2(x)\rho dx}{m}$. Examples: $f(x) = -x + 3$, $0 \le x \le 3$, quarter of circle, $f(x) = \sin x$, $0 \le x \le \pi$.

Volume, surface area of a body of revolution $V = \pi \int_a^b f^2(x)\, dx$, $F = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2} dx$. Example: sphere.

Lecture 22. (November 15, Friday)

Definition of improper integrals. Examples. (i) $\int_1^{+\infty} \frac{1}{x^2+1} dx = \frac{\pi}{2}$. (ii) $f(x) = \frac{1}{x}$, $1 \le x$. Divergent, the area is infinity, but the volume of the rotated figure is finite.

Approximate calculation of definite integrals by the help of the Maclaurin series of functions. Leibniz type series Example, calculate the integrals

(i) Error 0.1 $\int_0^1 e^{-x^2}\, dx = \int_0^1 (1 - x^2 + \frac{x^4}{2} - \ldots)dx = \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \ldots\right]_0^1 \approx 1 - \frac{1}{3} = \frac{2}{3}$.

(ii) until the cubic terms $\int_0^1 \frac{\sin x}{x} dx = \int_0^1 (1 - \frac{x^2}{6} + \frac{x^4}{120} - \ldots)dx = \left[x - \frac{x^3}{18} + \frac{x^5}{600} - \ldots\right]_0^1 \approx 1 - \frac{1}{18} = \frac{17}{18}$.

Week 12.

Lecture 23. (November 20, Wednesday).

Sample Test 2.

Lecture 24. (November 22, Friday) 8:15.
Test 2.

Week 13.

Lecture 25. (*November 27, Wednesday).*
First order ordinary differential equations, ODEs, $y' = \frac{dy}{dx} = f(x, y)$. Definition of the solution, integral curve, trajectory. Initial value (Cauchy) problem: $y' = f(x, y). y(x_0) = y_0$. Theorem of Existence & Uniqueness for the initial value problem (without proof) with conditions $f, \frac{\partial f}{\partial y} \in C^0(D)$.
Examples (guess the solutions). General solution. . Initial value problems $y' = y, y(0) = 1$, $y = e^x$. Maximal interval is "smaller".$y' = 1 + y^2, y(0) = 0, y = \tan x$. Checking the solution: $y' = 1 + x^2, y(0) = 0, y = x + \frac{x^3}{3}$.
Remark (in general): $y' = f(x)$, $y = \int f(x)dx + c$. There is no uniqueness $y' = 3y^{\frac{2}{3}}, y(0) = 0$.
Viewing of Test 2.

Lecture 26. (November 29, Friday).
University open day. No lecture.

Week 14.

Lecture 27. (December 4, Wednesday).
Retake 2. K. 364.

Lecture 28. (December 6, Friday).
Solution of a sample exam.
Viewing of Retake 2.
Summary of the course. Consultation.