Mathematics Global Exam, BME.
2020-06-16 in TEAMS. 12:15-13:45
100%. *Good Luck!*

The exercises depend on a parameter *D*. The parameter *D* is the **remainder** of the number of letters of **your full name in your NEPTUN code** divided by *8*. E.g. I am in NEPTUN as Dr Moson Peter, the number of letters is 12. 12=1*8+4, so *D=4*. Please **substitute** your *D* in the test and carry out the calculation with this value.

Please **upload your results** in form of **one pdf** (prepared by CamScanner) until **13:45** as the latest. Start the **title of your uploaded work with the parameter *D* (1 number) and continue with your NEPTUN code**.

The test starts on the next page.

*Theory (3*5=15%.).*
Name 3 **scientists**, who had important mathematical research related to our curriculum (Mathematics A1-A2-A3) and whose **surnames** (family names) start with one of the letters of **the last word in your NEPTUN code.** E.g. I am in NEPTUN as Dr Moson Peter, the last word is Peter, so my letters would be P, E, T, E, R. Write the main idea of **one mathematical result you met in our courses** (definition, theorem, etc.).

Points. Name 1%, and 1-4% depending on the explanation of the result. Altogether *3*5=15%*

Exercises 25+20+20+20=85%.

Please write all main steps of your solution.

Continuation is on the next page.

Notation: $[x]$ is called the **floor function** that takes as input a real number and gives as output the greatest integer less than or equal to **x**. Some examples $[3.9] = 3$, $[4] = 4$, $[4.9] = 4$, $[\pi] = 3$. **The next example contains the floor function.**

1.Let $\left[\boldsymbol{\beta} = \frac{D+2}{4}\right]$. (i) Find the general solution of the differential equation $y'' + 2y' + \boldsymbol{\beta} y = e^{\boldsymbol{\beta} x}$. (ii) Solve the solution of the initial value problem $y(0) = 0$, $y'(0) = 0$. (iii) Find the fourth order Maclaurin polynomial (Taylor's at $x_0 = 0$) of this solution by Newton's method.

10+5+5+5=25%

Continuation is on the next page.

Notation: $[x]$ is called the **floor function** that takes as input a real number and gives as output the greatest integer less than or equal to **x**. Some examples $[3.9] = 3$, $[4] = 4$, $[4.9] = 4$, $[\pi] = 3$. **The next example contains the floor function.**

2. Let $\boldsymbol{\delta}=\left[\frac{D}{2}\right]$. (i) Calculate the value of $z_1 = (1 + i\sqrt{3})^{3+\delta}$, $z_2 = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{6-2\delta}$, $z_3 = (D + 1)(e^{i\pi})^{\delta}$ (write the result in algebraic form). (ii) Find a lowest degree real polynomial (real coefficients) with roots z_1, z_2, z_3. (iii)What is the sum and product of the roots of this polynomial?
*3*4+4+2*2=20%*

Continuation is on the next page.

3. Consider the vector field $\vec{v}(\vec{r}) = (\vec{\imath} \times \vec{\jmath}) \times \vec{r}$, $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$. Calculate the value of the line integral $\int_\gamma \vec{v}(\vec{r})\, d\vec{r}$ by 2 different methods, if γ: $x = (D + 1)\cos t, y = \sin t,\ z = 0,\ 0 \le t \le 2\pi$.
5+10+5=20%

4. Consider the scalar filed $u(\vec{r}) = u(x, y, z) = (8 - D)x^2 + yz + \frac{z^3}{3}$. Let $\vec{v}(\vec{r}) = grad\ u(\vec{r}) = \nabla u(\vec{r})$.
(i) Calculate the value of the line integrals $\int_\gamma \vec{v}(\vec{r})\, d\vec{r}$, if $\gamma_{1,2}$ are the following curves γ_1: $x = (D + 1)\cos t, y = \sin t,\ z = 0,\ 0 \le t \le 2\pi$, és γ_2: $x = (D + 1)\cos t, y = \sin t,\ z = 0,\ 0 \le t \le \frac{\pi}{2}$.
(ii) Determine the value of the surface integral $\oiint_F \vec{v}(\vec{r})d\vec{F}$ if $F = \{(x, y, z)|x^2 + y^2 = 1, 0 \le z \le 1\}$, the normal points outward. *(5+5)+10=20%*

End.

Solutions (without figures)

*Theory (3*5=15%.).*
Points. Name 1%, and 1-4% depending on the explanation of the result. Altogether *3*5=15%*

Exercises 25+20+20+20=85%.

Second order linear equation. ***25%***
Homogeneous equation. $y'' + 2y' + \boldsymbol{\beta} y = e^{\boldsymbol{\beta} x}$.
Characteristic equation, general solution. $\left[\boldsymbol{\beta} = \frac{D+2}{4}\right]$
(i) $\boldsymbol{\beta} = 0, \mathrm{D} = 0, 1$: $y = c_1 e^{-2x} + c_2$, (ii) $\boldsymbol{\beta} = 1, \mathrm{D} = 2, 3, 4, 5$: $y = c_1 e^{-x} + c_2 x e^{-x}$, (iii) $\boldsymbol{\beta} = 2, \mathrm{D} = 6, 7$ $y = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$. *10%*
Non-homogeneous equation. Particular solution. (i) $\boldsymbol{\beta} = 0, \mathrm{D} = 0, 1$: $y_p(x) = x/2$, (ii) $\boldsymbol{\beta} = 1, \mathrm{D} = 2, 3, 4, 5$: $y_p(x) = e^x/4$, (iii) $\boldsymbol{\beta} = 2, \mathrm{D} = 6, 7$: $y_p(x) = e^{2x}/10$. *5%*
Initial value problem. (i) $\boldsymbol{\beta} = 0, \mathrm{D} = 0, 1$: $y = \frac{1}{4} e^{-2x} - \frac{1}{4}$+x/2 , (ii) $\boldsymbol{\beta} = 1, \mathrm{D} = 2, 3, 4, 5$: $y = -\frac{1}{4} e^{-x} - \frac{1}{2} x e^{-x}$+$e^x/4$, (iii) $\boldsymbol{\beta} = 2, \mathrm{D} = 6, 7$: $y = -\frac{1}{10} e^x \cos x - \frac{1}{10} e^x \sin x + e^{2x}/10$. *5%.*
Newton's method (fourth order terms): (i) $\boldsymbol{\beta} = 0, \mathrm{D} = 0, 1$: $y = \frac{1}{2} x^2 - \frac{1}{3} x^3 + \frac{1}{6} x^4$ (ii) $\boldsymbol{\beta} = 1, \mathrm{D} = 2, 3, 4, 5$: $y = \frac{1}{2} x^2 - \frac{1}{6} x^3 + \frac{1}{12} x^4$, (iii) $\boldsymbol{\beta} = 2, \mathrm{D} = 6, 7$: $y = \frac{1}{2} x^2 + \frac{1}{4} x^4$ 5*%.*

2. $\boldsymbol{\delta}=\left[\frac{D}{2}\right]$.

На $\boldsymbol{\delta}$=0, D=0. $z_1 = (1 + i\sqrt{3})^{3+\delta} = -8$, $z_2 = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{6-2\delta} = -i$, $z_3 = (D + 1)(e^{i\pi})^{\delta} = 1$.

Equation $(z + 8)(z^2 + 1)(z - 1)$. Roots. Sum -7, product -8.

На $\boldsymbol{\delta}$=0, D=1. $z_1 = (1 + i\sqrt{3})^{3+\delta} = -8$, $z_2 = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{6-2\delta} = -i$, $z_3 = (D + 1)(e^{i\pi})^{\delta} = 2$.

Equation $(z + 8)(z^2 + 1)(z - 2)$. Roots. Sum -6, product -16.

На $\boldsymbol{\delta}$=1, D=2. $z_1 = (1 + i\sqrt{3})^{3+\delta} = -8 - i8\sqrt{3}$, $z_2 = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{6-2\delta} = -1$, $z_3 = (D + 1)(e^{i\pi})^{\delta} = -3$.

Equation $(z^2 + 16z + 256)(z + 1)(z + 3)$. Roots. Sum -20, product 768.

На $\boldsymbol{\delta}$=1, D=3. $z_1 = (1 + i\sqrt{3})^{3+\delta} = -8 - i8\sqrt{3}$, $z_2 = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{6-2\delta} = -1$, $z_3 = (D + 1)(e^{i\pi})^{\delta} = -4$.

Equation $(z^2 + 16z + 256)(z + 1)(z + 4)$. Roots. Sum -21, product 1024.

If $\boldsymbol{\delta}$=2, D=4. $z_1 = (1 + i\sqrt{3})^{3+\delta} = 16 - i16\sqrt{3}$, $z_2 = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{6-2\delta} = i$, $z_3 = (D + 1)(e^{i\pi})^{\delta} = 5$.

Equation $(z^2 - 32z + 512)(z^2 + 1)(z - 5)$. Roots. Sum 37. Product 2560.

На $\boldsymbol{\delta}$=2, D=5. $z_1 = (1 + i\sqrt{3})^{3+\delta} = 16 - i16\sqrt{3}$, $z_2 = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{6-2\delta} = i$, $z_3 = (D + 1)(e^{i\pi})^{\delta} = 6$.

Equation $(z^2 - 32z + 1024)(z^2 + 1)(z - 6)$. Roots. Sum 38, product 6144.

*3*4+4+2*2=20%*

3. Stokes theorem. 20%.

The surface is an ellipse/circle in the plane z=0. Area $(D+1)\pi$.

$\vec{v}(\vec{r}) = (\vec{\imath} \times \vec{\jmath}) \times \vec{r} = \vec{k} \times \vec{r} = -y\vec{\imath} + x\vec{\jmath}$ $rot\vec{v}(\vec{r}) = 2\vec{k}$. Surface integral of the rotation $2(D+1)\pi$. Line integral by definition.

5+10+5=20%

4. Potential theory. Gauss-Ostrogradsky theorem. 20%.

$\vec{v}(\vec{r}) = grad\, u(\vec{r}) = \nabla\left((8-D)x^2 + yz + \frac{z^3}{3}\right) = (2(8-D)x)\vec{\imath} + z\vec{\jmath} + (y+z^2)\vec{k}$.

(i) Recommended method: use of the potential function. $\int_{\gamma_1} \vec{v}(\vec{r})\, d\vec{r} = 0$, the curve is closed. $\int_{\gamma_2} \vec{v}(\vec{r})\, d\vec{r} = u(0,1,0) - u(D+1,0,0)) = (D-8)(D+1)^2$. *5+5=10%*

(ii) Recommended method: use of the triple integral in the **Gauss-Ostrogradsky theorem**. $\vec{v}(\vec{r}) = (2(8-D)x)\vec{\imath} + z\vec{\jmath} + (y+z^2)\vec{k}$, $div\vec{v}(\vec{r}) = 2(8-D) + 2z$. $\oiint_F \vec{v}(\vec{r})d\vec{F} = 2(8-D)\pi + \pi$. *10%*