MATH EP1. TEST 1. October 16, 2020. 90 minutes. 20%. *Good Luck!*

1. Let $f(x) = \cos 2x \cos x$, $g(x) = \cosh 2x \cosh x$. (i) Find the 3rd order Maclaurin polynomial (Taylor at $x_0 = 0$) of the functions f, g. (ii) Calculate the limit $lim_{x\to 0} \frac{f(x)-g(x)}{x^2}$. *4%*
2. $OA = \vec{a} = 2\vec{\imath} + 2\vec{\jmath} + \vec{k} = (2,2,1)$, $OB = \vec{b} = \vec{\imath} - 2\vec{\jmath} + 2\vec{k} = (1,-2,2)$. Find $\vec{a} - \vec{b}$, $\vec{a} \cdot \vec{b}$, $\vec{a} \times \vec{b}$. How much is the perimeter, the area, the height passing through $0 = (0,0,0)$ of the triangle OAB ? *4%*
3. Consider the complex numbers: $z_1 = \sqrt{3} + i$, $z_2 = 2(\cos 45° + i\ \sin 45°)$. Find the value of the following complex numbers: (i) $z_1 - z_2$, (ii) $z_1 \cdot z_2$ (iii) $(z_1)^2$, (iv) $(z_2)^4$. *4%*
4. Find the equation of the tangent line to the function $f(x) = \arctan(2x)$ at the point $x_0 = \frac{1}{2}$ .Sketch the graph of the function and its tangent line. *4%*
5. Determine the type of local extrema (maximum, minimum), if they exist of the function $f(x) = x^2 e^{-2x}$. *4%*

SOLUTIONS (without figures)

1. (i) $f(x) = \cos 2x \cos x = (1 - 2x^2 + \cdots)\left(1 - \frac{x^2}{2} + \cdots\right) = 1 - \frac{5}{2}x^2 + \cdots$, $g(x) = \cosh 2x \cosh x = (1 + 2x^2 + \cdots)\left(1 + \frac{x^2}{2} + \cdots\right) = 1 + \frac{5}{2}x^2 + \cdots$ (ii) $lim_{x\to 0} \frac{f(x)-g(x)}{x^2} = -5$. *2*2=4%*

2. $\vec{a} - \vec{b} = (1,4,-1)$, $\vec{a} \cdot \vec{b} = 0$, $\vec{a} \times \vec{b} = (6,-3,-6)$. $Perimeter = 6 + 3\sqrt{2}$, Area $= \frac{9}{2}$, the height $\frac{3\sqrt{2}}{2}$. *4*0.5+2*1=4%*

3. $z_1 = \sqrt{3} + i = 2\ (\cos 30° + i\ \sin 30°)$, $z_2 = 2(\cos 45° + i\ \sin 45°) = \sqrt{2} + i\sqrt{2}$. $z_1 - z_2 = (\sqrt{3} - \sqrt{2}) + i(1 - \sqrt{2})$, $z_1 \cdot z_2 = 4\ (\cos 75° + i\ \sin 75°) = (\sqrt{6} - \sqrt{2}) + i\ (\sqrt{6} + \sqrt{2})$, $(z_1)^2 = 4\ (\cos 60° + i\ \sin 60°) = 2 + i\ 2\sqrt{3}$, $(z_2)^4 = 16(\cos 180° + i\ \sin 180°) = -16$. *2*0.5+3*1=4%*

4. The equation of the tangent line $y - f(x_0) = f'(x_0) \cdot (x - x_0)$. Here $x_0 = \frac{1}{2}$, $f(x_0) = \arctan 1 = \frac{\pi}{4}$, $f'(x) = \frac{2}{1+4x^2}$, $f'\left(\frac{1}{2}\right) = 1$. $y - \frac{\pi}{4} = \left(x - \frac{1}{2}\right)$. *3+1=4%*

5. $f'(x) = 2e^{-2x}(x - x^2)$. $f'(x) = 0$ if $x_0 = 0, x_1 = 1$. $f''(x) = 2e^{-2x}(1 - 4x + 2x^2)$. $f''(0) = 2 > 0$, $x_0 = 0$ is a minimum, $f''(1) = -2e^{-2} < 0$, $x_1 = 1$ is a local maximum. *2*2=4%*